A SCHOOL GEOMETRY

PARTS I.–IV.

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BY

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AND

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is section of the work is the position of I. 2

The examples are numerous and for the most part easy. They have been very carefully arranged, and are distributed throughout the text in immediate connection with the propositions on which they depend. A special feature is the large number of examples involving graphical or numerical work. The answers to these have been printed on perforated pages, so that they may easily be removed if it is found that access to numerical results is a source of temptation in examples involving measurement.

We are indebted to several friends for advice and suggestions. In particular we wish to express our thanks to Mr. H. C. Playne and Mr. H. C. Beaven of Clifton College for the valuable assistance they have rendered in reading the proof sheets and checking the answers to some of the numerical exercises.

H. S. HALL.
F. H. STEVENS.

November, 1903.

PREFATORY NOTE TO THE SECOND EDITION.

In the present edition some further steps have been taken towards the curtailment of bookwork by reducing certain less important propositions (e.g. Euclid I. 22, 43, 44) to the rank of exercises. Room has thus been found for more numerical and graphical exercises, and experimental work such as that leading to the Theorem of Pythagoras.

Theorem 22 (page 62), in the shape recommended in the Cambridge Schedule, replaces the equivalent proposition given as *Additional Theorem A* (page 60) in previous editions.

In the case of a few problems (e.g. Problems 23, 28, 29) it has been thought more instructive to justify the construction by a preliminary *analysis* than by the usual formal proof.

H. S. HALL.
F. H. STEVENS.

March, 1904.

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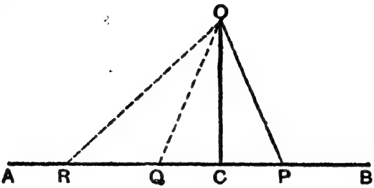
PART II.

PART IV.

Theorem 12.

Of all straight lines drawn from a given point to a given straight line the perpendicular is the least.



Let OC be the perpendicular, and OP *any* oblique, drawn from he given point O to the given straight line AB.

It is required to prove that OC *is less than* OP.

Proof. In the $\triangle$ OCP, since the $\angle$ OCP is a right angle,

$\therefore$ the $\angle$ OPC is less than a right angle; *Theor.* 8. *Cor.*

hat is, the $\angle$ OPC is less than the $\angle$ OCP.

$\therefore$ OC is less than OP. *Theor.* 10.

Q.E.D.

Corollary 1. Hence conversely, since there can be only ne perpendicular and one shortest line from O to AB,

If OC *is the shortest straight line from* O *to* AB, *then* OC *is* *erpendicular to* AB.

Corollary 2. *Two obliques* OP, OQ, *which cut* AB *at equal istances from* C *the foot of the perpendicular, are equal.*

The $\triangle^s$ OCP, OCQ may be shewn to be congruent by Theorem 4; hence OP = OQ.

Corollary 3. *Of two obliques* OQ, OR, *if* OR *cuts* AB *at the eater distance from* C *the foot of the perpendicular, then* OR *is eater than* OQ.

The $\angle$OQC is acute, $\therefore$ the $\angle$OQR is obtuse;

$\therefore$ the $\angle$OQR is greater than the $\angle$ORQ;

$\therefore$ OR is greater than OQ.

EXERCISES ON INEQUALITIES IN A TRIANGLE.

1. *The hypotenuse is the greatest side of a right-angled triangle.*

2. *The greatest side of any triangle makes acute angles with each of the other sides.*

3. *If from the ends of a side of a triangle, two straight lines are drawn to a point within the triangle, then these straight lines are together less than the other two sides of the triangle.*

4. BC, the base of an isosceles triangle ABC, is produced to any point D; shew that AD is greater than either of the equal sides.

5. If in a quadrilateral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle.

6. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A and terminated by the base BC, is less than AB.

7. ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively: shew that, if AB is greater than AC, then OB is greater than OC.

8. The difference of any two sides of a triangle is less than the third side.

9. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

10. The perimeter of a quadrilateral is greater than the sum of its diagonals.

11. ABC is a triangle, and the vertical angle BAC is bisected by a line which meets BC in X; shew that BA is greater than BX, and CA greater than CX. Hence obtain a proof of Theorem 11.

12. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

13. The sum of the diagonals of a quadrilateral is less than the sum of the four straight lines drawn from the angular points to any given point. Prove this, and point out the exceptional case.

14. *In a triangle any two sides are together greater than twice the median which bisects the remaining side.*

[Produce the median, and complete the construction after the manner of Theorem 8.]

15. *In any triangle the sum of the medians is less than the perimeter.*

PARALLELS.

DEFINITION. **Parallel straight lines** are such as, being in the same plane, do not meet however far they are produced beyond both ends.

NOTE. Parallel lines must be *in the same plane.* For instance, two straight lines, one of which is drawn on a table and the other on the floor would never meet if produced; but they are not for that reason necessarily parallel.

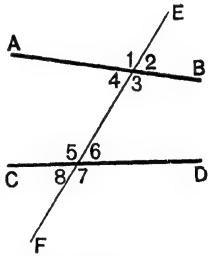
AXIOM. *Two intersecting straight lines cannot both be parallel to a third straight line.*

In other words:

Through a given point there can be only one straight line parallel to a given straight line.

This assumption is known as *Playfair's Axiom.*

DEFINITION. When two straight lines AB, CD are met by a third straight line EF, *eight* angles are formed, to which for the sake of distinction particular names are given.

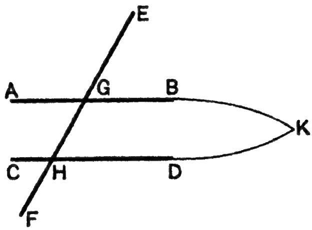


Thus in the adjoining figure,
1, 2, 7, 8 are called **exterior** angles,
3, 4, 5, 6 are called **interior** angles,
4 and 6 are said to be **alternate** angles; so also the angles 3 and 5 are alternate to one another.

Of the angles 2 and 6, 2 is referred to as the **exterior** angle, and 6 as the **interior opposite** angle on the same side of EF. Such angles are also known as **corresponding angles.** Similarly 7 and 3, 8 and 4, 1 and 5 are pairs of corresponding angles.

THEOREM 13. [Euclid I. 27 and 28.]

If a straight line cuts two other straight lines so as to make
(i) *the alternate angles equal,*
or (ii) *an exterior angle equal to the interior opposite angle on the same side of the cutting line,*
or (iii) *the interior angles on the same side equal to two right angles;*
then in each case the two straight lines are parallel.



(i) Let the straight line EGHF cut the two straight lines AB, CD at G and H so as to make the alternate ∠s AGH, GHD equal to one another.

It is required to prove that AB *and* CD *are parallel.*

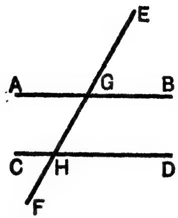
Proof. If AB and CD are not parallel, they will meet, if produced, either towards B and D, or towards A and C.

If possible, let AB and CD, when produced, meet towards B and D, at the point K.

Then KGH is a triangle, of which one side KG is produced to A;
∴ the exterior ∠AGH is greater than the interior opposite ∠GHK; but, by hypothesis, it is not greater.

∴ AB and CD cannot meet when produced towards B and D.
Similarly it may be shewn that they cannot meet towards A and C:

∴ AB and CD are parallel.



(ii) Let the exterior ∠ EGB = the interior opposite ∠ GHD.

It is required to prove that AB *and* CD *are parallel.*

Proof. Because the ∠ EGB = the ∠ GHD,
and the ∠ EGB = the vertically opposite ∠ AGH;
∴ the ∠ AGH = the ∠ GHD:
and these are alternate angles;
∴ AB and CD are parallel.

(iii) Let the two interior ∠^{s} BGH, GHD be together equal to two right angles.

It is required to prove that AB *and* CD *are parallel.*

Proof. Because the ∠^{s} BGH, GHD together = two right angles;
and because the adjacent ∠^{s} BGH, AGH together = two right angles;
∴ the ∠^{s} BGH, AGH together = the ∠^{s} BGH, GHD.
From these equals take the ∠ BGH;
then the remaining ∠ AGH = the remaining ∠ GHD:
and these are alternate angles;
∴ AB and CD are parallel.

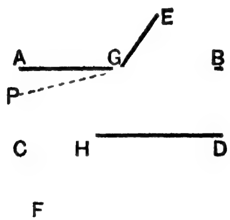
Q.E.D.

DEFINITION. A straight line drawn across a set of given lines is called a **transversal**.

For instance, in the above diagram the line EGHF, which crosses the given lines AB, CD is a transversal.

THEOREM 14. [Euclid I. 29.]

If a straight line cuts two parallel lines, it makes
(i) *the alternate angles equal to one another;*
(ii) *the exterior angle equal to the interior opposite angle on the same side of the cutting line;*
(iii) *the two interior angles on the same side together equal to two right angles.*



Let the straight lines AB, CD be parallel, and let the straight line EGHF cut them.

It is required to prove that

(i) *the* ∠ AGH = *the alternate* ∠ GHD;
(ii) *the exterior* ∠ EGB = *the interior opposite* ∠ GHD;
(iii) *the two interior* ∠ˢ BGH, GHD *together* = *two right angles.*

Proof. (i) If the ∠ AGH is not equal to the ∠ GHD, suppose the ∠ PGH equal to the ∠ GHD, and alternate to it;
then PG and CD are parallel. *Theor.* 13.

But, by hypothesis, AB and CD are parallel;
∴ the two intersecting straight lines AG, PG are both parallel to CD: which is impossible. *Playfair's Axiom.*

∴ the ∠ AGH is not unequal to the ∠ GHD;
that is, the alternate ∠ˢ AGH, GHD are equal.

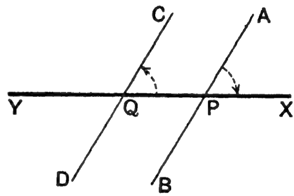
(ii) Again, because the ∠ EGB = the vertically opposite ∠ AGH;
and the ∠ AGH = the alternate ∠ GHD; *Proved.*
∴ the exterior ∠ EGB = the interior opposite ∠ GHD.

(iii) Lastly, the $\angle$ EGB = the $\angle$ GHD ; *Proved.*
add to each the $\angle$ BGH ;
then the $\angle^s$ EGB, BGH together = the angles BGH, GHD.
But the adjacent $\angle^s$ EGB, BGH together = two right angles ;
$\therefore$ the two interior $\angle^s$ BGH, GHD together = two right angles.
Q.E.D.

PARALLELS ILLUSTRATED BY ROTATION.

The **direction** of a straight line is determined by the angle which it makes with some given line of reference.

Thus the direction of AB, relatively to the given line YX, is given by the angle APX.



Now suppose that AB and CD in the adjoining diagram are parallel; then we have learned that
the ext. $\angle$APX = the int. opp. $\angle$CQX;
that is, AB and CD *make equal angles* with the line of reference YX.

This brings us to the leading idea connected with parallels :
Parallel straight lines have the same DIRECTION, *but differ in* POSITION.

The same idea may be illustrated thus :

Suppose AB to rotate about P through the $\angle$APX, so as to take the position XY. Thence let it rotate about Q *the opposite way* through the equal $\angle$XQC : it will now take the position CD. Thus AB may be brought into the position of CD by two rotations which, being equal and opposite, involve no final change of direction.

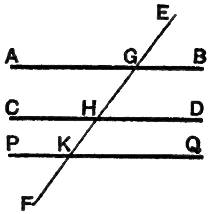
HYPOTHETICAL CONSTRUCTION. In the above diagram let AB be a fixed straight line, Q a fixed point, CD a straight line turning about Q, and YQPX any transversal through Q. Then as CD rotates, there must be one position in which the $\angle$CQX = the fixed $\angle$APX.

Hence through any given point we may assume a line to pass parallel to any given straight line.

Obs. If AB is a straight line, movements from A towards B, and from B towards A are said to be in opposite **senses** of the line AB.

THEOREM 15. [Euclid I. 30.]

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to the straight line PQ.

It is required to prove that AB *and* CD *are parallel to one another.*

Draw a straight line EF cutting AB, CD, and PQ in the points G, H, and K.

Proof. Then because AB and PQ are parallel, and EF meets them,

$\therefore$ the $\angle$AGK = the alternate $\angle$GKQ.

And because CD and PQ are parallel, and EF meets them,
$\therefore$ the exterior $\angle$GHD = the interior opposite $\angle$GKQ.

$\therefore$ the $\angle$AGH = the $\angle$GHD;
and these are alternate angles;
$\therefore$ AB and CD are parallel.

Q.E.D.

NOTE. If PQ lies between AB and CD, the Proposition needs no proof; for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another.

The truth of this Proposition may be readily deduced from Playfair's Axiom, of which it is the converse.

For if AB and CD were not parallel, they would meet when produced. Then there would be two intersecting straight lines both parallel to a third straight line: which is impossible.

Therefore AB and CD never meet; that is, they are parallel.

EXERCISES ON PARALLELS.

1. In the diagram of the previous page, if the angle EGB is 55°, express in degrees each of the angles GHC, HKQ, QKF.

2. *Straight lines which are perpendicular to the same straight line are parallel to one another.*

3. *If a straight line meet two or more parallel straight lines, and is perpendicular to one of them, it is also perpendicular to all the others.*

4. *Angles of which the arms are parallel, each to each, are either equal or supplementary.*

5. Two straight lines AB, CD bisect one another at O. Shew that the straight lines joining AC and BD are parallel.

6. Any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

7. If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, the triangle thus formed is isosceles.

8. From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z: shew the triangle AYZ is isosceles.

9. If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, shew that the triangle is isosceles.

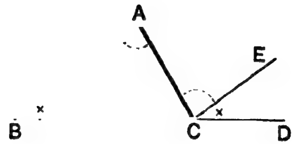
10. The straight lines drawn from any point in the bisector of an angle parallel to the arms of the angle, and terminated by them, are equal; and the resulting figure is a rhombus.

11. AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected: if through any point X in DC a straight line YXZ is drawn parallel to AB and meeting the bisectors in Y and Z, shew that XY is equal to XZ.

12. Two straight rods PA, QB revolve about pivots at P and Q, PA making 12 complete revolutions a minute, and QB making 10. If they start parallel and pointing the same way, how long will it be before they are again parallel, (i) pointing opposite ways, (ii) pointing the same way?

THEOREM 16. [Euclid I. 32.]

The three angles of a triangle are together equal to two right angles.



Let ABC be a triangle.

It is required to prove that the three $\angle^s$ ABC, BCA, CAB *together* = *two right angles.*

Produce BC to any point D; and suppose CE to be the line through C parallel to BA.

Proof. Because BA and CE are parallel and AC meets them,
∴ the $\angle$ ACE = the alternate $\angle$ CAB.

Again, because BA and CE are parallel, and BD meets them.
∴ the exterior $\angle$ ECD = the interior opposite $\angle$ ABC.

∴ *the whole exterior* $\angle$ ACD = *the sum of the two interior opposite* $\angle^s$ CAB, ABC.

To each of these equals add the $\angle$ BCA;
then the $\angle^s$ BCA, ACD together = the three $\angle^s$ BCA, CAB, ABC.

But the adjacent $\angle^s$ BCA, ACD together = two right angles
∴ the $\angle^s$ BCA, CAB, ABC together = two right angles.

Q.E.D.

Obs. In the course of this proof the following most important property has been established.

If a side of a triangle is produced the exterior angle is equal to the sum of the two interior opposite angles.

Namely, the ext. $\angle$ ACD = the $\angle$ CAB + the $\angle$ ABC.

INFERENCES FROM THEOREM 16.

1. *If* A, B, *and* C *denote the number of degrees in the angles of a triangle,*

$$\textit{then } A + B + C = 180°.$$

2. *If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to the third angle of the other.*

3. *In any right-angled triangle the two acute angles are complementary.*

4. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

5. *The sum of the angles of any quadrilateral figure is equal to four right angles.*

EXERCISES ON THEOREM 16.

1. Each angle of an equilateral triangle is two-thirds of a right angle, or 60°.

2. In a right-angled isosceles triangle each of the equal angles is 45°.

3. Two angles of a triangle are 36° and 123° respectively: deduce the third angle; and verify your result by measurement.

4. In a triangle ABC, the $\angle B = 111°$, the $\angle C = 42°$; deduce the $\angle A$, and verify by measurement.

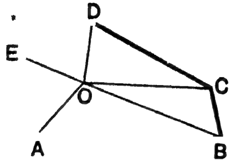
5. One side BC of a triangle ABC is produced to D. If the exterior angle ACD is 134°; and the angle BAC is 42°; find each of the remaining interior angles.

6. In the figure of Theorem 16, if the $\angle ACD = 118°$, and the $\angle B = 51°$, find the $\angle$s A and C; and check your results by measurement.

7. *Prove that the three angles of a triangle are together equal to two right angles by supposing a line drawn through the vertex parallel to the base.*

8. *If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair.*

COROLLARY 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



Let ABCDE be a rectilineal figure of n sides.

It is required to prove that all the interior angles + 4 rt. $\angle^s$ $= 2n$ rt. $\angle^s$.

Take any point O within the figure, and join O to each of its vertices.

Then the figure is divided into n triangles.

And the three $\angle^s$ of each $\triangle$ together $= 2$ rt. $\angle^s$.
Hence *all* the $\angle^s$ of *all* the $\triangle^s$ together $= 2n$ rt. $\angle^s$.

But all the $\angle^s$ of all the $\triangle^s$ make up all the interior angles of the figure together with the angles at O, which $= 4$ rt. $\angle^s$.

$\therefore$ all the int. $\angle^s$ of the figure + 4 rt. $\angle^s = 2n$ rt. $\angle^s$.

Q.E.D.

DEFINITION. A **regular** polygon is one which has all its sides equal and all its angles equal.

Thus if D denotes the number of degrees in each angle of a *regular* polygon of n sides, the above result may be stated thus:

$$n\mathrm{D} + 360^\circ = n \, . \, 180^\circ.$$

EXAMPLE.

Find the number of degrees in each angle of

(i) a regular hexagon (6 sides);
(ii) a regular octagon (8 sides);
(iii) a regular decagon (10 sides).

EXERCISES ON THEOREM 16.

(Numerical and Graphical.)

1. ABC is a triangle in which the angles at B and C are respectively double and treble of the angle at A: find the number of degrees in each of these angles.

2. Express in degrees the angles of an isosceles triangle in which
(i) Each base angle is double of the vertical angle;
(ii) Each base angle is four times the vertical angle.

3. The base of a triangle is produced both ways, and the exterior angles are found to be 94° and 126°; deduce the vertical angle. Construct such a triangle, and check your result by measurement.

4. The sum of the angles at the base of a triangle is 162°, and their difference is 60°: find all the angles.

5. The angles at the base of a triangle are 84° and 62°; deduce (i) the vertical angle (ii) the angle between the bisectors of the base angles. Check your results by construction and measurement.

6. In a triangle ABC, the angles at B and C are 74° and 62°; if AB and AC are produced, deduce the angle between the bisectors of the exterior angles. Check your result graphically.

7. Three angles of a quadrilateral are respectively $114\frac{1}{2}°$, 50°, and $75\frac{1}{2}°$; find the fourth angle.

8. In a quadrilateral ABCD, the angles at B, C, and D are respectively equal to 2A, 3A, and 4A; find all the angles.

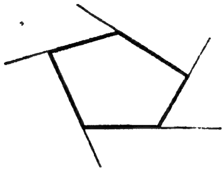
9. Four angles of an irregular pentagon (5 sides) are 40°, 78°, 122°, and 135°; find the fifth angle.

10. In any *regular* polygon of n sides, each angle contains $\frac{2(n-2)}{n}$ right angles.
(i) Deduce this result from the Enunciation of Corollary 1.
(ii) Prove it independently by joining one vertex A to each of the others (except the two immediately adjacent to A), thus dividing the polygon into $n-2$ triangles.

11. How many sides have the regular polygons each of whose angles is (i) 108°, (ii) 156°?

12. Shew that the only *regular* figures which may be fitted together so as to form a plane surface are (i) *equilateral triangles*, (ii) *squares*, (iii) *regular hexagons*.

COROLLARY 2. *If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.*



1st Proof. Suppose, as before, that the figure has n sides; and consequently n vertices.

Now at *each* vertex

$$\text{the interior } \angle + \text{the exterior } \angle = 2 \text{ rt. } \angle^s;$$

and there are n vertices,

$$\therefore \text{ the sum of the int. } \angle^s + \text{the sum of the ext. } \angle^s = 2n \text{ rt. } \angle^s.$$

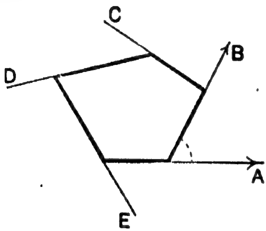
But by Corollary 1,

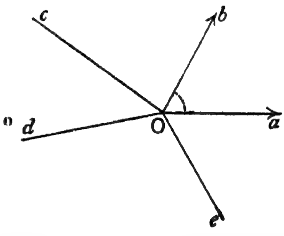
$$\text{the sum of the int. } \angle^s + \quad 4 \text{ rt. } \angle^s \quad = 2n \text{ rt. } \angle^s;$$

$$\therefore \text{ the sum of the ext. } \angle^s = 4 \text{ rt. } \angle^s.$$

Q.E.D

2nd Proof.





Take any point O, and suppose Oa, Ob, Oc, Od, and Oe, are lines parallel to the sides marked, A, B, C, D, E (and drawn from O in the *sense* in which those sides were produced).

Then the exterior $\angle$ between the sides A and B = the $\angle\, a\text{O}b$.

And the other exterior $\angle^s$ = the $\angle^s$ bOc, cOd, dOe, eOa, respectively.

$$\therefore \text{ the sum of the ext. } \angle^s = \text{the sum of the } \angle^s \text{ at O}$$
$$= 4 \text{ rt. } \angle^s.$$

EXERCISES.

1. If one side of a regular hexagon is produced, shew that the exterior angle is equal to the interior angle of an equilateral triangle.

2. Express in degrees the magnitude of each *exterior* angle of (i) a regular octagon, (ii) a regular decagon.

3. How many sides has a regular polygon if each exterior angle is (i) 30°, (ii) 24°?

4. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected, shew that the bisectors meet at right angles.

5. If the base of any triangle is produced both ways, shew that the sum of the two exterior angles *minus* the vertical angle is equal to two right angles.

6. In the triangle ABC the base angles at B and C are bisected by BO and CO respectively. Shew that the angle $BOC = 90° + \frac{A}{2}$.

7. In the triangle ABC, the sides AB, AC are produced, and the exterior angles are bisected by BO and CO. Shew that the angle $BOC = 90° - \frac{A}{2}$.

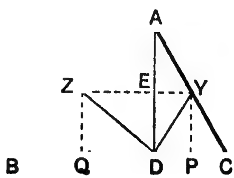
8. The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles.

9. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA; if DC is drawn, shew that BCD is a right angle.

10. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

EXPERIMENTAL PROOF OF THEOREM 16. [$A+B+C=180°$.]

In the $\triangle$ ABC, AD is perp. to BC the greatest side. AD is bisected at right angles by ZY; and YP, ZQ are perp^s^. on BC.

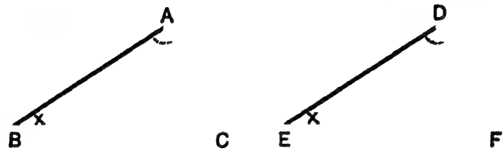


If now the $\triangle$ is folded about the three dotted lines, the $\angle$^s^ A, B, and C will coincide with the $\angle$^s^ ZDY, ZDQ, YDP;

$\therefore$ their sum is 180°.

THEOREM 17. [Euclid I. 26.]

If two triangles have two angles of one equal to two angles of the other, each to each, and any side of the first equal to the corresponding side of the other, the triangles are equal in all respects.



Let ABC, DEF be two triangles in which
the $\angle$ A = the $\angle$ D,
the $\angle$ B = the $\angle$ E,
also let the side BC = the corresponding side EF.

It is required to prove that the $\triangle^s$ ABC, DEF *are equal in all respects.*

Proof. The sum of the $\angle^s$ A, B, and C
= 2 rt. $\angle^s$ *Theor.* 16.
= the sum of the $\angle^s$ D, E, and F;
and the $\angle^s$ A and B = the $\angle^s$ D and E respectively,
$\therefore$ the $\angle$ C = the $\angle$ F.

Apply the $\triangle$ ABC to the $\triangle$ DEF, so that B falls on E, and BC along EF.

Then because BC = EF,
$\therefore$ C must coincide with F.

And because the $\angle$ B = the $\angle$ E,
$\therefore$ BA must fall along ED.

And because the $\angle$ C = the $\angle$ F,
$\therefore$ CA must fall along FD.

$\therefore$ the point A, which falls both on ED and on FD, must coincide with D, the point in which these lines intersect.

$\therefore$ the $\triangle$ ABC coincides with the $\triangle$ DEF,
and is therefore equal to it in all respects.
So that AB = DE, and AC = DF;
and the $\triangle$ ABC = the $\triangle$ DEF in area. Q.E.D.

EXERCISES.

ON THE IDENTICAL EQUALITY OF TRIANGLES.

1. Shew that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

2. *Any point on the bisector of an angle is equidistant from the arms of the angle.*

3. Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are dropped upon it from A and B: shew that AX is equal to BY.

4. If the bisector of the vertical angle of a triangle is at right angles to the base, the triangle is isosceles.

5. If in a triangle the perpendicular from the vertex on the base bisects the base, then the triangle is isosceles.

6. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

[Produce the bisector, and complete the construction after the manner of Theorem 8.]

7. The middle point of any straight line which meets two parallel straight lines, and is terminated by them, is equidistant from the parallels.

8. A straight line drawn between two parallels and terminated by them, is bisected; shew that any other straight line passing through the middle point and terminated by the parallels, is also bisected at that point.

9. If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal.

10. In a quadrilateral, ABCD, if AB=AD, and BC=DC: shew that the diagonal AC bisects each of the angles which it joins; and that AC is perpendicular to BD.

11. *A surveyor wishes to ascertain the breadth of a river which he cannot cross. Standing at a point* A *near the bank, he notes an object* B *immediately opposite on the other bank. He lays down a line* AC *of any length at right angles to* AB, *fixing a mark at* O *the middle point of* AC. *From* C *he walks along a line perpendicular to* AC *until he reaches a point* D *from which* O *and* B *are seen in the same direction. He now measures* CD: *prove that the result gives him the width of the river.*

ON THE IDENTICAL EQUALITY OF TRIANGLES.

Three cases of the congruence of triangles have been dealt with in Theorems 4, 7, 17, the results of which may be summarised as follows:

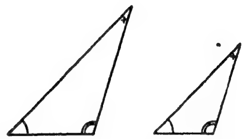
Two triangles are equal in all respects when the following three parts in each are severally equal:

1. *Two sides, and the included angle.* Theorem 4.
2. *The three sides.* Theorem 7
3. *Two angles and one side, the side given in one triangle* CORRESPONDING *to that given in the other.* Theorem 17.

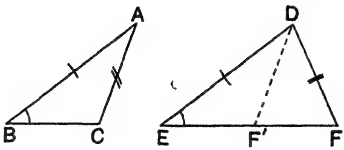
Two triangles are not, however, necessarily equal in all respects when *any three parts* of one are equal to the corresponding parts of the other.

For example:

(i) When the *three angles* of one are equal to the *three angles* of the other, each to each, the adjoining diagram shews that the triangles need not be equal in all respects.



(ii) When *two sides and one angle* in one are equal to *two sides and one angle* of the other, the given angles being *opposite* to equal sides, the diagram below shews that the triangles need not be equal in all respects.



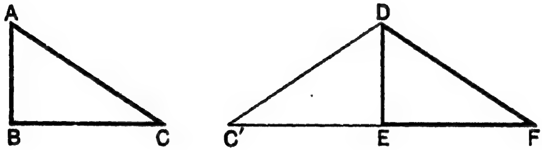
For if AB = DE, and AC = DF, and the ∠ABC = the ∠DEF, it will be seen that the shorter of the given sides in the triangle DEF may lie in either of the positions DF or DF′.

NOTE. From these data it may be shewn that the angles opposite to the equal sides AB, DE are either *equal* (as for instance the ∠ˢ ACB, DF′E) or *supplementary* (as the ∠ˢ ACB, DFE); and that in the former case the triangles are equal in all respects. This is called the **ambiguous case** in the congruence of triangles. [See Problem 9, p. 82.]

If the given angles at B and E are right-angles, the ambiguity disappears. This exception is proved in the following Theorem.

THEOREM 18.

Two **right-angled** *triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are equal in all respects.*



Let ABC, DEF be two **right-angled** triangles, in which
the $\angle^s$ ABC, DEF are right angles,
the hypotenuse AC = the hypotenuse DF,
and AB = DE.

It is required to prove that the $\triangle^s$ ABC, DEF *are equal in all respects.*

Proof. Apply the $\triangle$ ABC to the $\triangle$ DEF, so that AB falls on the equal line DE, and C on the side of DE opposite to F.

Let C′ be the point on which C falls.

Then DEC′ represents the $\triangle$ ABC in its new position.

Since each of the $\angle^s$ DEF, DEC′ is a right angle,
∴ EF and EC′ are in one straight line.

And in the $\triangle$ C′DF, because DF = DC′ (*i.e.* AC),
∴ the $\angle$ DFC′ = the $\angle$ DC′F. *Theor.* 5.

Hence in the $\triangle^s$ DEF, DEC′,

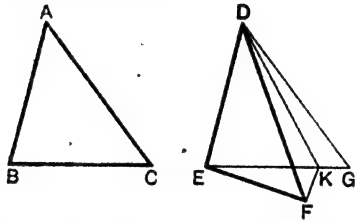
because the $\angle$ DEF = the $\angle$ DEC′, being right angles;
the $\angle$ DFE = the $\angle$ DC′E, *Proved.*
and the side DE is common

∴ the $\triangle^s$ DEF, DEC′ are equal in all respects; *Theor.* 17,
that is, the $\triangle^s$ DEF, ABC are equal in all respects.

Q.E.D.

*Theorem 19. [Euclid I. 24.]

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle included by the two sides of one greater than the angle included by the corresponding sides of the other; then the base of that which has the greater angle is greater than the base of the other.



Let ABC, DEF be two triangles, in which

BA = ED,

and AC = DF,

but the ∠ BAC is greater than the ∠ EDF.

It is required to prove that the base BC *is greater than the base* EF.

Proof. Apply the △ ABC to the △ DEF, so that A falls on D, and AB along DE.

Then because AB = DE, B must coincide with E.

Let DG, GE represent AC, CB in their new position.

Then if EG passes through F, EG is greater than EF;

that is, BC is greater than EF.

But if EG does not pass through F, suppose that DK bisects the ∠ FDG, and meets EG in K. Join FK.

Then in the △s FDK, GDK,

because $\begin{cases} \text{FD} = \text{GD}, \\ \text{DK is common to both,} \\ \text{and the included } \angle\, \text{FDK} = \text{the included } \angle\, \text{GDK}; \end{cases}$

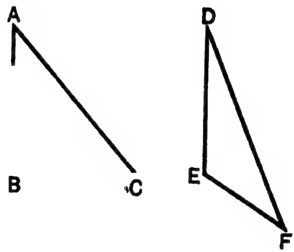
∴ FK = GK. *Theor.* 4.

Now the two sides EK, KF are greater than EF;

that is, EK, KG are greater than EF.

∴ EG (or BC) is greater than EF. Q.E.D.

Conversely, *if two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; then the angle contained by the sides of that which has the greater base, is greater than the angle contained by the corresponding sides of the other.*



Let ABC, DEF be two triangles in which

BA = ED,

and AC = DF,

but the base BC is greater than the base EF.

It is required to prove that the ∠ BAC *is greater than the* ∠ EDF.

Proof. If the ∠ BAC is not greater than the ∠ EDF,
it must be either equal to, or less than the ∠ EDF.

Now if the ∠ BAC were equal to the ∠ EDF,
then the base BC would be equal to the base EF; *Theor.* 4.
but, by hypothesis, it is not.

Again, if the ∠ BAC were less than the ∠ EDF,
then the base BC would be less than the base EF; *Theor.* 19.
but, by hypothesis, it is not.

That is, the ∠ BAC is neither equal to, nor less than the ∠ EDF;
∴ the ∠ BAC is greater than the ∠ EDF.

Q.E.D.

* *Theorems marked with an asterisk may be omitted or postponed at the discretion of the teacher.*

REVISION LESSON ON TRIANGLES.

1. State the properties of a triangle relating to

(i) the sum of its interior angles;

(ii) the sum of its exterior angles.

What property corresponds to (i) in a polygon of n sides? With what other figures does a triangle share the property (ii)?

2. Classify triangles with regard to their angles. Enunciate any Theorem or Corollary assumed in the classification.

3. Enunciate two Theorems in which from data relating to the sides a conclusion is drawn relating to the angles.

In the triangle ABC, if $a=3{\cdot}6$ cm., $b=2{\cdot}8$ cm., $c=3{\cdot}6$ cm., arrange the angles in order of their sizes (before measurement); and prove that the triangle is acute-angled.

4. Enunciate two Theorems in which from data relating to the angles a conclusion is drawn relating to the sides.

In the triangle ABC, if

(i) $A=48°$ and $B=51°$, find the third angle, and name the greatest side.

(ii) $A=B=62\frac{1}{2}°$, find the third angle, and arrange the sides in order of their lengths.

5. From which of the conditions given below may we conclude that the triangles ABC, A′B′C′ are identically equal? Point out where ambiguity arises; and draw the triangle ABC in each case.

(i) $\begin{cases} A=A'=71°. \\ B=B'=46°. \\ a=a'=3{\cdot}7 \text{ cm.} \end{cases}$ (ii) $\begin{cases} a=a'=4{\cdot}2 \text{ cm.} \\ b=b'=2{\cdot}4 \text{ cm.} \\ C=C'=81°. \end{cases}$ (iii) $\begin{cases} A=A'=36°. \\ B=B'=121°. \\ C=C'=23°. \end{cases}$

(iv) $\begin{cases} a=a'=3{\cdot}0 \text{ cm.} \\ b=b'=5{\cdot}2 \text{ cm.} \\ c=c'=4{\cdot}5 \text{ cm.} \end{cases}$ (v) $\begin{cases} B=B'=53°. \\ b=b'=4{\cdot}3 \text{ cm.} \\ c=c'=5{\cdot}0 \text{ cm.} \end{cases}$ (vi) $\begin{cases} C=C'=90°. \\ c=c'=5 \text{ cm.} \\ a=a'=3 \text{ cm.} \end{cases}$

6. Summarise the results of the last question by stating generally under what conditions two triangles

(i) are necessarily congruent;

(ii) may or may not be congruent.

7. *If two triangles have their angles equal, each to each, the triangles are not necessarily equal in all respects, because the three data are not* **independent.** Carefully explain this statement.

(*Miscellaneous Examples.*)

8. (i) *The perpendicular is the shortest line that can be drawn to a given straight line from a given point.*

(ii) **Obliques** *which make equal angles with the perpendicular are equal.*

(iii) *Of two obliques the less is that which makes the smaller angle with the perpendicular.*

9. *If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are equal in all respects.*

10. PQ is a perpendicular (4 cm. in length) to a straight line XY. Draw through P a series of obliques making with PQ the angles 15°, 30°, 45°, 60°, 75°. *Measure* the lengths of these obliques, and tabulate the results.

11. PAB is a triangle in which AB and AP have constant lengths 4 cm. and 3 cm. If AB is fixed, and AP rotates about A, trace the changes in PB, as the angle A increases from 0° to 180°.

Answer this question by drawing a series of figures, increasing A by increments of 30°. Measure PB in each case, and tabulate the results.

12. From B the foot of a flagstaff AB a horizontal line is drawn passing two points C and D which are 27 feet apart. The angles BCA and BDA are 65° and 40° respectively. Represent this on a diagram (scale 1 cm. to 10 ft.), and find by measurement the approximate height of the flagstaff.

13. From P, the top of a lighthouse PQ, two boats A and B are seen at anchor in a line due south of the lighthouse. It is known that PQ=126 ft., $\angle$PAQ=57°, $\angle$PBQ=33°; hence draw a plan in which 1″ represents 100 ft., and find by measurement the distance between A and B to the nearest foot.

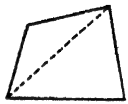
14. From a lighthouse L two ships A and B, which are 600 yards apart, are observed in directions S.W. and 15° East of South respectively. At the same time B is observed from A in a S.E. direction. Draw a plan (scale 1″ to 200 yds.), and find by measurement the distance of the lighthouse from each ship.

PARALLELOGRAMS.

DEFINITIONS.

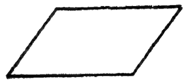
1. A **quadrilateral** is a plane figure bounded by *four* straight lines.

The straight line which joins opposite angular points in a quadrilateral is called a **diagonal**.



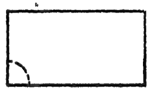
2. A **parallelogram** is a quadrilateral whose opposite sides are *parallel*.

[It will be proved hereafter that the opposite sides of a parallelogram are equal, and that its opposite angles are equal.]



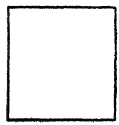
3. A **rectangle** is a parallelogram which has one of its angles a right angle.

[It will be proved hereafter that *all* the angles of a rectangle are right angles. See page 59.]

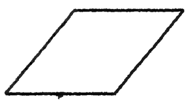


4. A **square** is a rectangle which has two adjacent sides equal.

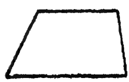
[It will be proved that *all* the sides of a square are equal and all its angles right angles. See page 59.]



5. A **rhombus** is a quadrilateral which has all its sides equal, but its angles are not right angles.

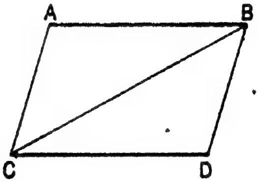


6. A **trapezium** is a quadrilateral which has *one* pair of parallel sides.



THEOREM 20. [Euclid I. 33.]

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB and CD be equal and parallel straight lines; and let them be joined towards the same parts by the straight lines AC and BD.

It is required to prove that AC *and* BD *are equal and parallel.*

Join BC.

Proof. Then because AB and CD are parallel, and BC meets them,

$\therefore$ the $\angle$ ABC = the alternate $\angle$ DCB.

Now in the $\triangle^s$ ABC, DCB,

because $\begin{cases} \text{AB} = \text{DC}, \\ \text{BC is common to both;} \\ \text{and the } \angle \text{ ABC} = \text{the } \angle \text{ DCB;} \end{cases}$ *Proved.*

$\therefore$ the triangles are equal in all respects;

so that AC = DB,..........................(i)

and the $\angle$ ACB = $\angle$ DBC.

But these are alternate angles;

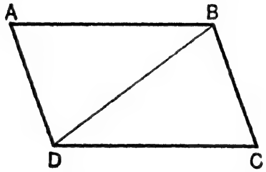
$\therefore$ AC and BD are parallel. ....(ii)

That is, AC and BD are both equal and parallel.

Q.E.D.

THEOREM 21 [Euclid I. 34.]

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram.



Let ABCD be a parallelogram, of which BD is a diagonal.

It is required to prove that

(i) AB = CD, *and* AD = CB,
(ii) *the* ∠ BAD = *the* ∠ DCB,
(iii) *the* ∠ ADC = *the* ∠ CBA,
(iv) *the* △ ABD = *the* △ CDB *in area.*

Proof. Because AB and DC are parallel, and BD meets them,
∴ the ∠ ABD = the alternate ∠ CDB.

Again, because AD and BC are parallel, and BD meets them,
∴ the ∠ ADB = the alternate ∠ CBD.

Hence in the △ˢ ABD, CDB,

because { the ∠ ABD = the ∠ CDB,
the ∠ ADB = the ∠ CBD, *Proved.*
and BD is common to both ;

∴ the triangles are equal in all respects ; *Theor.* 17.
so that AB = CD, and AD = CB ; .... (i)
and the ∠ BAD = the ∠ DCB ; .... (ii)
and the △ ABD = the △ CDB in area. ....(iv)

And because the ∠ ADB = the ∠ CBD, *Proved.*
and the ∠ CDB = the ∠ ABD,
∴ the whole ∠ ADC = the whole ∠ CBA.........(iii)

Q.E.D.

COROLLARY 1. *If one angle of a parallelogram is a right le, all its angles are right angles.*

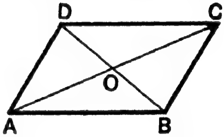
n other words:

All *the angles of a rectangle are right angles.*

For the sum of two consecutive $\angle^s = 2$ rt. $\angle^s$; (*Theor.* 14.)
$\therefore$, if one of these is a rt. angle, the other must be a rt. angle.
And the opposite angles of the par^m are equal;
$\therefore$ all the angles are rt. angles.

COROLLARY 2. *All the sides of a square are equal; and all s angles are right angles.*

COROLLARY 3. *The diagonals of a parallelogram bisect one nother.*



Let the diagonals AC, BD of the par^m BCD intersect at O.

To prove AO = OC, *and* BO = OD.

In the $\triangle^s$ AOB, COD,

$$\text{because}\begin{cases}\text{the } \angle OAB = \text{the alt. } \angle OCD,\\ \text{the } \angle AOB = \text{vert. opp. } \angle COD,\\ \quad\text{and } AB = \text{the opp. side } CD;\end{cases}$$

$\therefore$ OA = OC; and OB = OD. *Theor.* 17.

EXERCISES.

1. *If the opposite sides of a quadrilateral are equal, the figure is a rallelogram.*

2. *If the opposite angles of a quadrilateral are equal, the figure is a rallelogram.*

3. *If the diagonals of a quadrilateral bisect each other, the figure is a rallelogram.*

4. *The diagonals of a rhombus bisect one another at right angles.*

5. If the diagonals of a parallelogram are equal, all its angles are ht angles.

6. In a parallelogram which is not rectangular the diagonals are qual.

EXERCISES ON PARALLELS AND PARALLELOGRAMS.

(*Symmetry and Superposition.*)

1. Shew that by folding a rhombus about one of its diagonals the triangles on opposite sides of the crease may be made to coincide.

That is to say, prove that a rhombus is *symmetrical* about either diagonal.

2. Prove that the diagonals of a square are *axes of symmetry*. Name two other lines about which a square is symmetrical.

3. The diagonals of a rectangle divide the figure into two congruent triangles: is the diagonal, therefore, an axis of symmetry? About what two lines is a rectangle symmetrical?

4. Is there any axis about which an oblique parallelogram is symmetrical? Give reasons for your answer.

5. In a quadrilateral ABCD, AB=AD and CB=CD; but the sides are not all equal. Which of the diagonals (if either) is an axis of symmetry?

6. Prove by the method of superposition that

(i) *Two parallelograms are identically equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each, and one angle of one equal to one angle of the other.*

(ii) *Two rectangles are equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each.*

7. Two quadrilaterals ABCD, EFGH have the sides AB, BC, CD, DA equal respectively to the sides EF, FG, GH, HE, and have also the angle BAD equal to the angle FEH. Shew that the figures may be made to coincide with one another.

(*Miscellaneous Theoretical Examples.*)

8. Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point.

9. In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal.

10. If ABCD is a parallelogram, and X, Y respectively the middle points of the sides AD, BC; shew that the figure AYCX is a parallelogram.

11. ABC and DEF are two triangles such that AB, BC are respectively equal to and parallel to DE, EF; shew that AC is equal and parallel to DF.

12. ABCD is a quadrilateral in which AB is parallel to DC, and AD equal but not parallel to BC; shew that

(i) the $\angle A +$ the $\angle C = 180° =$ the $\angle B +$ the $\angle D$;

(ii) the diagonal AC = the diagonal BD;

(iii) the quadrilateral is *symmetrical* about the straight line joining the middle points of AB and DC.

13. AP, BQ are straight rods of equal length, turning at equal rates (both clockwise) about two fixed pivots A and B respectively. If the rods start parallel but pointing in opposite senses, shew that

(i) they will always be parallel;

(ii) the line joining PQ will always pass through a certain fixed point.

(*Miscellaneous Numerical and Graphical Examples.*)

14. Calculate the angles of the triangle ABC, having given

$$\text{int. } \angle A = \tfrac{3}{7} \text{ of ext. } \angle A; \quad 3B = 4C.$$

15. A yacht sailing due East changes her course successively by 63°, by 78°, by 119°, and by 64°, with a view to sailing round an island. What further change must be made to set her once more on an Easterly course?

16. If the sum of the interior angles of a rectilineal figure is equal to the sum of the exterior angles, how many sides has it, and why?

17. Draw, using your protractor, any five-sided figure ABCDE, in which

$$\angle B = 110°, \quad \angle C = 115°, \quad \angle D = 93°, \quad \angle E = 152°.$$

Verify by a construction with ruler and compasses that AE is parallel to BC, and account theoretically for this fact.

18. A and B are two fixed points, and two straight lines AP, BQ, unlimited towards P and Q, are pivoted at A and B. AP, starting from the direction AB, turns about A clockwise at the uniform rate of $7\tfrac{1}{2}°$ a second; and BQ, starting simultaneously from the direction BA, turns about B counter-clockwise at the rate of $3\tfrac{3}{4}°$ a second.

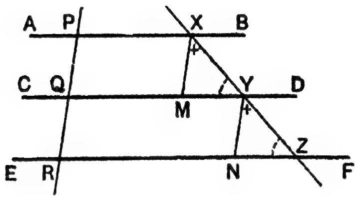
(i) How many seconds will elapse before AP and BQ are parallel?

(ii) Find graphically and by calculation the angle between AP and BQ twelve seconds from the start.

(iii) At what rate does this angle decrease?

THEOREM 22.

If there are three or more parallel straight lines, and the intercepts made by them on any transversal are equal, then the corresponding intercepts on any other transversal are also equal.



Let the parallels AB, CD, EF cut off equal intercepts PQ, QR from the transversal PQR; and let XY, YZ be the corresponding intercepts cut off from any other transversal XYZ.

It is required to prove that XY = YZ.

Through X and Y let XM and YN be drawn parallel to PR.

Proof. Since CD and EF are parallel, and XZ meets them,

∴ the $\angle$ XYM = the corresponding $\angle$ YZN.

And since XM, YN are parallel, each being parallel to PR,

∴ the $\angle$ MXY = the corresponding $\angle$ NYZ.

Now the figures PM, QN are parallelograms,

∴ XM = the opp. side PQ; and YN = the opp. side QR;

and since by hypothesis PQ = QR,

∴ XM = YN.

Then in the $\triangle$s XMY, YNZ,

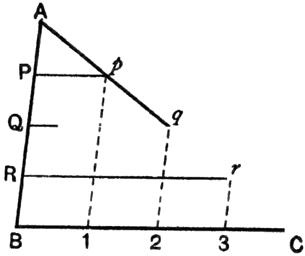
because $\begin{cases} \text{the } \angle \text{XYM} = \text{the } \angle \text{YZN}, \\ \text{the } \angle \text{MXY} = \text{the } \angle \text{NYZ}, \\ \text{and XM} = \text{YN}; \end{cases}$

∴ the triangles are identically equal; *Theor.* 17.

∴ XY = YZ.

Q.E.D.

COROLLARY. *In a triangle* ABC, *if a set of lines* Pp, Qq, Rr, ..., *drawn parallel to the base, divide one side* AB *into equal parts, they also divide the other side* AC *into equal parts.*



The lengths of the parallels Pp, Qq, Rr, ..., may thus be expressed in terms of the base BC.

Through p, q, and r let $p1$, $q2$, $r3$ be drawn par$^{\text{l}}$ to AB.

Then, by Theorem 22, these par$^{\text{ls}}$ divide BC into *four* equal parts, of which Pp evidently contains *one*, Qq *two*, and Rr *three*.

In other words,

$$Pp=\frac{1}{4}\cdot BC; \qquad Qq=\frac{2}{4}\cdot BC; \qquad Rr=\frac{3}{4}\cdot BC.$$

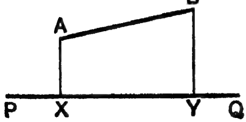
Similarly if the given par$^{\text{ls}}$ divide AB into n equal parts,

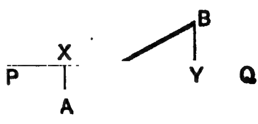
$$Pp=\frac{1}{n}\cdot BC, \qquad Qq=\frac{2}{n}\cdot BC, \qquad Rr=\frac{3}{n}\cdot BC; \text{ and so on.}$$

*** *Problem* 7, *p.* 78, *should now be worked.*

DEFINITION.

If from the extremities of a straight line AB perpendiculars AX, BY are drawn to a straight line PQ of indefinite length, then XY is said to be the **orthogonal projection** of AB on PQ.





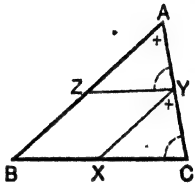
EXERCISES ON PARALLELS AND PARALLELOGRAMS.

1. *The straight line drawn through the middle point of a side of a triangle, parallel to the base, bisects the remaining side.*

[This is an important particular case of Theorem 22.

In the $\triangle ABC$, if Z is the middle point of AB, and ZY is drawn par[1] to BC, we have to prove that $AY = YC$.

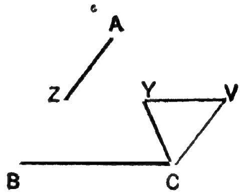
Draw YX par[1] to AB, and then prove the $\triangle^s$ ZAY, XYC congruent.]



2. *The straight line which joins the middle points of two sides of a triangle is parallel to the third side.*

[In the $\triangle ABC$, if Z, Y are the middle points of AB, AC, we have to prove ZY par[1] to BC.

Produce ZY to V, making YV equal to ZY, and join CV. Prove the $\triangle^s$ AYZ, CYV congruent; the rest follows at once.]



3. *The straight line which joins the middle points of two sides of a triangle is equal to half the third side.*

4. *Shew that the three straight lines which join the middle points of the sides of a triangle, divide it into four triangles which are identically equal.*

5. *Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.*

6. ABCD is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC: shew that BX and DY trisect the diagonal AC.

7 *If the middle points of adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram.*

8. Shew that the straight lines which join the middle points of opposite sides of a quadrilateral, bisect one another.

9. From two points A and B, and from O the mid-point between them, perpendiculars AP, BQ, OX are drawn to a straight line CD. If AP, BQ measure respectively 4·2 cm. and 5·8 cm., *deduce* the length of OX, and verify your result by measurement.

Shew that $OX = \frac{1}{2}(AP + BQ)$ or $\frac{1}{2}(AP \sim BQ)$, according as A and B are on the *same* side, or on *opposite* sides of CD.

10. When three parallels cut off equal intercepts from two transversals, shew that of the three parallel lengths between the two transversals the middle one is the **Arithmetic Mean** of the other two.

11. *The parallel sides of a trapezium are* a *centimetres and* b *centimetres in length. Prove that the line joining the middle points of the oblique sides is parallel to the parallel sides, and that its length is* $\frac{1}{2}(a+b)$ *centimetres.*

12. OX and OY are two straight lines, and along OX five points 1, 2, 3, 4, 5 are marked at equal distances. Through these points parallels are drawn in any direction to meet OY. Measure the lengths of these parallels: take their average, and compare it with the length of the *third* parallel. Prove *geometrically* that the 3rd parallel is the mean of all five.

State the corresponding theorem for any odd number $(2n+1)$ of parallels so drawn.

13. From the angular points of a parallelogram perpendiculars are drawn to any straight line which is outside the parallelogram: shew that the sum of the perpendiculars drawn from one pair of opposite angles is equal to the sum of those drawn from the other pair.

[Draw the diagonals, and from their point of intersection suppose a perpendicular drawn to the given straight line.]

14. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides is equal to the perpendicular drawn from either extremity of the base to the opposite side.

[It follows that the sum of the distances of *any* point in the base of an isosceles triangle from the equal sides is **constant**, that is, the same whatever point in the base is taken.]

How would this property be modified if the given point were taken in the base *produced*?

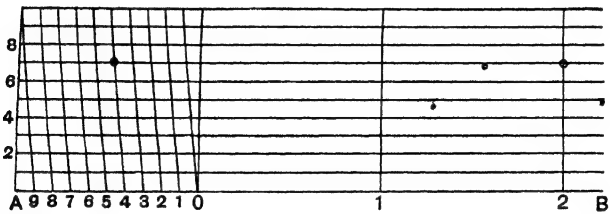
15. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the perpendicular drawn from any one of the angular points to the opposite side, and is therefore constant.

16. Equal and parallel lines have equal projections on any other straight line.

DIAGONAL SCALES.

Diagonal scales form an important application of Theorem 22 We shall illustrate their construction and use by describing a *Decimal Diagonal Scale to shew Inches, Tenths, and Hundredths.*

A straight line AB is divided (from A) into inches, and the points of division marked 0, 1, 2 .... The primary division 0A is subdivided into *tenths,* these secondary divisions being numbered (from 0) 1, 2, 3, ... 9. We may now read on AB *inches* and *tenths* of an inch.



In order to read *hundredths,* ten lines are taken at any equal intervals parallel to AB; and perpendiculars are drawn through 0, 1, 2, ...

The primary (or inch) division corresponding to 0A on the tenth parallel is now subdivided into *ten* equal parts; and diagonal lines are drawn, as in the diagram,

joining 0 to the *first* point of subdivision on the 10th parallel.
" 1 to the *second* " " " "
" 2 to the *third* " " " "
and so on.

The scale is now complete, and its use is shewn in the following example.

Example. To take from the scale a length of 2·47 *inches.*

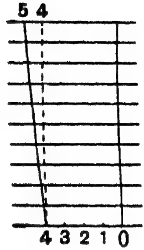
(i) Place one point of the dividers at 2 in AB, and extend them till the other point reaches 4 in the subdivided inch 0A. We have now 2·4 inches in the dividers.

(ii) To get the remaining 7 *hundredths,* move the right-hand point up the perpendicular through 2 till it reaches the 7th parallel. Then extend the dividers till the left point reaches the diagonal 4 also on the 7th parallel. We have now 2·47 inches in the dividers.

REASON FOR THE ABOVE PROCESS.

The first step needs no explanation. The reason of the second is found in the Corollary of Theorem 22.

Joining the point **4** to the corresponding point on the tenth parallel, we have a triangle **4,4,5**, of which one side **4,4** is divided into ten equal parts by a set of lines parallel to the base **4,5**.



Therefore the lengths of the parallels between **4,4**, and the diagonal **4,5** are $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, ... of the base, which is ·1 inch.

Hence these lengths are respectively

·01, ·02, ·03, ... of 1 inch.

Similarly, by means of this scale, the length of a given straight line may be measured to the nearest hundredth of an inch.

Again, if one inch-division on the scale is taken to represent 10 feet, then 2·47 inches on the scale will represent 24·7 feet. And if one inch-division on the scale represents 100 links, then 2·47 inches will represent 247 links. Thus a diagonal scale is of service in preparing plans of enclosures, buildings, or field-works, where it is necessary that every dimension of the actual object must be represented by a line of proportional length on the plan.

NOTE.

The subdivision of a diagonal scale need not be *decimal*.

For instance we might construct a diagonal scale to read centimetres, millimetres, and *quarters* of a millimetre; in which case we should take *four* parallels to the line AB.

[For Exercises on Linear Measurements see the following page.]

EXERCISES ON LINEAR MEASUREMENTS.

1. Draw straight lines whose lengths are 1·25 inches, 2·72 inches, 3·08 inches.

2. Draw a line 2·68 inches long, and measure its length in centimetres and the nearest millimetre.

3. Draw a line 5·7 cm. in length, and measure it in inches (to the nearest hundredth). Check your result by calculation, given that 1 cm. = 0·3937 inch.

4. Find by measurement the equivalent of 3·15 inches in centimetres and millimetres. Hence calculate (correct to two decimal places) the value of 1 cm. in inches.

5. Draw lines 2·9 cm. and 6·2 cm. in length, and measure them in inches. Use each equivalent to find the value of 1 inch in centimetres and millimetres, and take the average of your results.

6. A distance of 100 miles is represented on a map by 1 inch. Draw lines to represent distances of 336 miles and 408 miles.

7. If 1 inch on a map represents 1 kilometre, draw lines to represent 850 metres, 2980 metres, and 1010 metres.

8. A plan is drawn to the scale of 1 inch to 100 links. Measure in centimetres and millimetres a line representing 417 links.

9. Find to the nearest hundredth of an inch the length of a line which will represent 42·500 kilometres in a map drawn to the scale of 1 centimetre to 5 kilometres.

10. The distance from London to Oxford (in a direct line) is 55 miles. If this distance is represented on a map by 2·75 inches, to what scale is the map drawn? That is, how many miles will be represented by 1 inch? How many kilometres by 1 centimetre?
[1 cm. = 0·3937 inch; 1 km. = $\frac{5}{8}$ mile, nearly.]

11. On a map of France drawn to the scale 1 inch to 35 miles, the distance from Paris to Calais is represented by 4·2 inches. Find the distance accurately in miles, and approximately in kilometres, and express the scale in metric measure. [1 km. = $\frac{5}{8}$ mile, nearly.]

12. The distance from Exeter to Plymouth is $37\frac{1}{2}$ miles, and appears on a certain map to be $2\frac{1}{2}''$; and the distance from Lincoln to York is 88 km., and appears on another map to be 7 cm. Compare the scales of these maps in miles to the inch.

13. Draw a diagonal scale, 2 centimetres to represent 1 yard, shewing yards, feet, and inches.

PRACTICAL GEOMETRY.

PROBLEMS.

The following problems are to be solved with ruler and compasses only. No step requires the actual measurement of any line or angle; that is to say, the *constructions* are to be made without using either a graduated scale of length, or a protractor.

The problems are not merely to be studied as propositions; but the construction in every case is to be actually performed by the learner, great care being given to accuracy of drawing.

Each problem is followed by a *theoretical* proof; but the *results* of the work should always be verified by measurement, as a test of correct drawing. Accurate measurement is also required in applications of the problems.

In the diagrams of the problems lines which are inserted only for purposes of *proof* are dotted, to distinguish them from lines necessary to the construction.

For practical applications of the problems the student should be provided with the following instruments:

1. A flat ruler, one edge being graduated in centimetres and millimetres, and the other in inches and tenths.

2. Two set squares; one with angles of 45°, and the other with angles of 60° and 30°.

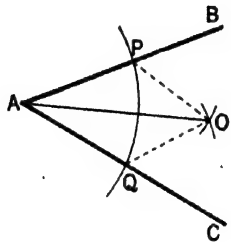
3. A pair of pencil compasses.

4. A pair of dividers, preferably with screw adjustment.

5. A semi-circular protractor.

Problem 1.

To bisect a given angle.



Let BAC be the given angle to be bisected

Construction. With centre A, and any radius, draw an arc of a circle cutting AB, AC at P and Q.

With centres P and Q, and radius PQ, draw two arcs cutting at O. Join AO.

Then the $\angle$ BAC is bisected by AO.

Proof. Join PO, QO.

In the $\triangle$s APQ, AQO,

because $\begin{cases} AP = AQ, \text{ being radii of a circle,} \\ PO = QO, \quad ,, \quad ,, \quad \text{equal circles,} \\ \text{and } AO \text{ is common;} \end{cases}$

$\therefore$ the triangles are equal in all respects; *Theor.* 7.

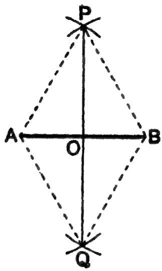
so that the $\angle$ PAO = the $\angle$ QAO;

that is, the $\angle$ BAC is bisected by AO.

Note. PQ has been taken as the radius of the arcs drawn from the centres P and Q, and the intersection of these arcs determines the point O. *Any* radius, however, may be used instead of PQ, provided that it is great enough to secure the intersection of the arcs.

PROBLEM 2.

To bisect a given straight line.



Let AB be the line to be bisected.

Construction. With centre A, and radius AB, draw two arcs, one on each side of AB.

With centre B, and radius BA, draw two arcs, one on each side of AB, cutting the first arcs at P and Q.

Join PQ, cutting AB at O.

Then AB is bisected at O

Proof. Join AP, AQ, BP, BQ.

In the $\triangle^s$ APQ, BPQ,

because $\begin{cases} AP = BP, \text{ being radii of equal circles,} \\ AQ = BQ, \text{ for the same reason,} \\ \text{and } PQ \text{ is common;} \end{cases}$

$\therefore$ the $\angle APQ =$ the $\angle BPQ$. *Theor.* 7.

Again in the $\triangle^s$ APO, BPO,

because $\begin{cases} AP = BP, \\ PO \text{ is common,} \\ \text{and the } \angle APO = \text{the } \angle BPO; \end{cases}$

$\therefore$ AO = OB; *Theor.* 4.

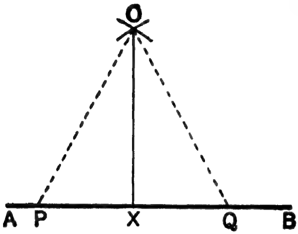
that is, AB is bisected at O.

NOTES. (i) AB was taken as the radius of the arcs drawn from the centres A and B, but *any* radius may be used provided that it is great enough to secure the intersection of the arcs which determine the points P and Q.

(ii) From the congruence of the $\triangle^s$ APO, BPO it follows that the $\angle AOP =$ the $\angle BOP$. As these are adjacent angles, it follows that PQ *bisects* AB *at right angles.*

PROBLEM 3.

To draw a straight line perpendicular to a given straight line at a given point in it.



Let AB be the straight line, and X the point in it at which a perpendicular is to be drawn.

Construction. With centre X cut off from AB any two equal parts XP, XQ.

With centres P and Q, and radius PQ, draw two arcs cutting at O.

Join XO.

Then XO is perp. to AB.

Proof. Join OP, OQ.

In the $\triangle^s$ OXP, OXQ,

because $\begin{cases} XP = XQ, \text{ by construction,} \\ OX \text{ is common,} \\ \text{and } PO = QO, \text{ being radii of equal circles;} \end{cases}$

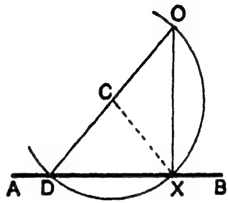
$\therefore$ the $\angle OXP$ = the $\angle OXQ$. *Theor.* 7.

And these being adjacent angles, each is a right angle;

that is, XO is perp. to AB.

Obs. If the point X is near one end of AB, one or other of the alternative constructions on the next page should be used.

PROBLEM 3. SECOND METHOD.



Construction. Take any point C outside AB.

With centre C, and radius CX, draw a circle cutting AB at D.

Join DC, and produce it to meet the circumference of the circle at O.

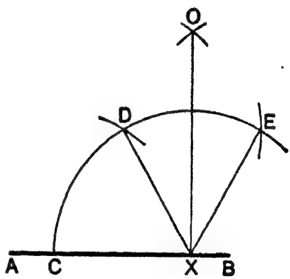
Join XO.

Then XO is perp. to AB.

Proof. Join CX.

Because CO = CX; ∴ the ∠ CXO = the ∠ COX;
and because CD = CX; ∴ the ∠ CXD = the ∠ CDX.
∴ the whole ∠ DXO = the ∠ XOD + the ∠ XDO
$= \frac{1}{2}$ of 180°
= 90°.
∴ XO is perp. to AB.

PROBLEM 3. THIRD METHOD.



Construction. With centre X and any radius, draw the arc CDE, cutting AB at C.

With centre C, and with the same radius, draw an arc, cutting the first arc at D.

With centre D, and with the same radius, draw an arc cutting the first arc at E.

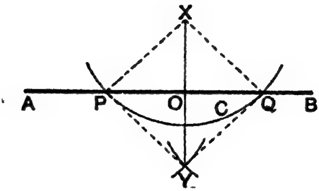
Bisect the ∠ DXE by XO. *Prob.* 1.

Then XO is perp. to AB.

Proof. Each of the ∠s CXD, DXE may be proved to be 60°;
and the ∠ DXO is half of the ∠ DXE;
∴ the ∠ CXO is 90°.
That is, XO is perp. to AB.

PROBLEM 4.

To draw a straight line perpendicular to a given straight line from a given external point.



Let X be the given external point from which a perpendicular is to be drawn to AB.

Construction. Take any point C on the side of AB remote from X.

With centre X, and radius XC, draw an arc to cut AB at P and Q.

With centres P and Q, and radius PX, draw arcs cutting at Y, on the side of AB opposite to X.

Join XY cutting AB at O.

Then XO is perp. to AB.

Proof. Join PX, QX, PY, QY,

In the $\triangle^s$ PXY, QXY,

$$\text{because}\left\{\begin{array}{l}\text{PX} = \text{QX, being radii of a circle,}\\ \text{PY} = \text{QY, for the same reason,}\\ \text{and XY is common;}\end{array}\right.$$

$\therefore$ the $\angle$ PXY = the $\angle$ QXY. *Theor.* 7.

Again, in the $\triangle^s$ PXO, QXO,

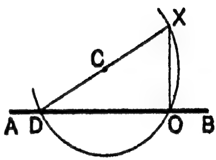
$$\text{because}\left\{\begin{array}{l}\text{PX} = \text{QX,}\\ \text{XO is common,}\\ \text{and the } \angle\,\text{PXO} = \text{the } \angle\,\text{QXO;}\end{array}\right.$$

$\therefore$ the $\angle$ XOP = the $\angle$ XOQ. *Theor.* 4.

And these being adjacent angles, each is a right angle, that is, XO is perp. to AB.

Obs. When the point X is nearly opposite one end of AB, one or other of the alternative constructions given below should be used.

PROBLEM 4. SECOND METHOD.



Construction. Take any point D in AB. Join DX, and bisect it at C.

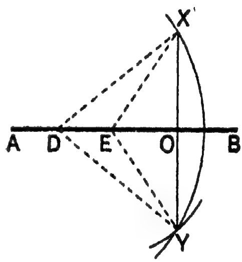
With centre C, and radius CX, draw a circle cutting AB at D and O.

Join XO.

Then XO is perp. to AB.

For, as in Problem 3, Second Method, the ∠XOD is a right angle.

PROBLEM 4. THIRD METHOD.



Construction. Take any two points D and E in AB.

With centre D, and radius DX, draw an arc of a circle, on the side of AB opposite to X.

With centre E, and radius EX, draw another arc cutting the former at Y.

Join XY, cutting AB at O.

Then XO is perp. to AB.

(i) Prove the △ˢ XDE, YDE equal in all respects by Theorem 7,

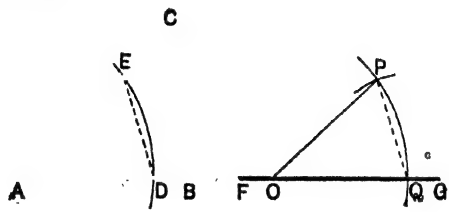
so that the ∠XDE = the ∠YDE.

(ii) Hence prove the △ˢ XDO, YDO equal in all respects by Theorem 4, so that the adjacent ∠ˢ DOX, DOY are equal.

That is, XO is perp. to AB.

Problem 5.

At a given point in a given straight line to make an angle equal to a given angle.



Let BAC be the given angle, and FG the given straight line; and let O be the point at which an angle is to be made equal to the $\angle$ BAC.

Construction. With centre A, and with any radius, draw an arc cutting AB and AC at D and E.

With centre O, and with the same radius, draw an arc cutting FG at Q.

With centre Q, and with radius DE, draw an arc cutting the former arc at P.

Join OP.

Then POQ is the required angle.

Proof. Join ED, PQ.

In the $\triangle^{s}$ POQ, EAD,

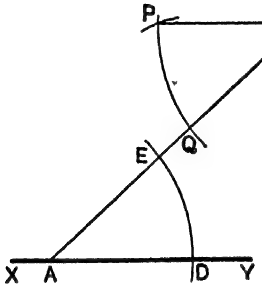
because $\begin{cases} \text{OP} = \text{AE, being radii of equal circles,} \\ \text{OQ} = \text{AD, for the same reason,} \\ \text{PQ} = \text{ED, by construction;} \end{cases}$

$\therefore$ the triangles are equal in all respects;

so that the $\angle$ POQ = the $\angle$ EAD. *Theor.* 7.

PROBLEM 6.

Through a given point to draw a straight line parallel to a given straight line.



Let XY be the given straight line, and O the given point, through which a straight line is to be drawn par[1] to XY.

Construction. In XY take any point A, and join OA.

Using the construction of Problem 5, at the point O in the line AO make the ∠AOP equal to the ∠OAY and alternate to it.

Then OP is parallel to XY.

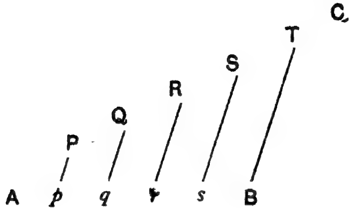
Proof. Because AO, meeting the straight lines OP, XY, makes the alternate ∠s POA, OAY equal;

∴ OP is par[1] to XY.

*** *The constructions of Problems* 3, 4, *and* 6 *are not usually followed in practical applications. Parallels and perpendiculars may be more quickly drawn by the aid of set squares.* (*See* LESSONS IN EXPERIMENTAL GEOMETRY, *pp.* 36, 42.)

Problem 7.

To divide a given straight line into any number of equal parts.



Let AB be the given straight line, and suppose it is required to divide it into *five* equal parts.

Construction. From A draw AC, a straight line of unlimited length, making any angle with AB.

From AC mark off *five* equal parts of *any* length, AP, PQ, QR, RS, ST.

Join TB; and through P, Q, R, S draw par[ls] to TB, meeting AB in *p*, *q*, *r*, *s*.

Then since the par[ls] P*p*, Q*q*, R*r*, S*s*, TB cut off five equal parts from AT, they also cut off five equal parts from AB. (Theorem 22.)

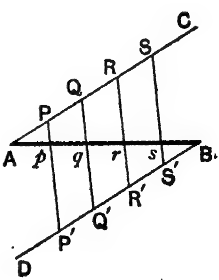
Second Method.

From A draw AC at any angle with AB, and on it mark off *four* equal parts AP, PQ, QR, RS, of any length.

From B draw BD par[l] to AC, and on it mark off BS′, S′R′, R′Q′, Q′P′, each equal to the parts marked on AC.

Join PP′, QQ′, RR′, SS′ meeting AB in *p*, *q*, *r*, *s*. Then AB is divided into five equal parts at these points.

[Prove by Theorems 20 and 22.]



EXERCISES ON LINES AND ANGLES.

(Graphical Exercises.)

1. Construct (with ruler and compasses only) an angle of 60°.
By repeated bisection divide this angle into *four* equal parts.

2. By means of Exercise 1, *trisect a right angle*; that is, divide it into *three* equal parts.

Bisect each part, and hence shew how to trisect an angle of 45°.
[No construction is known for exactly trisecting *any* angle.]

3. Draw a line 6·7 cm. long, and divide it into *five* equal parts. Measure one of the parts in inches (to the nearest hundredth), and verify your work by calculation. [1 cm. = 0·3937 inch.]

4. From a straight line 3·72″ long, cut off *one seventh*. Measure the part in centimetres and the nearest millimetre, and verify your work by calculation.

5. At a point X in a straight line AB draw XP perpendicular to AB, making XP 1·8″ in length. From P draw an oblique PQ, 3·0″ long, to meet AB in Q. Measure XQ.

(Problems. State your construction, and give a theoretical proof.)

6. In a straight line XY find a point which is equidistant from two given points A and B.
When is this impossible?

7. In a straight line XY find a point which is equidistant from two intersecting lines AB, AC.
When is this impossible?

8. From a given point P draw a straight line PQ, making with a given straight line AB an angle of given magnitude.

9. From two given points P and Q on the same side of a straight line AB, draw two lines which meet in AB and make equal angles with it.

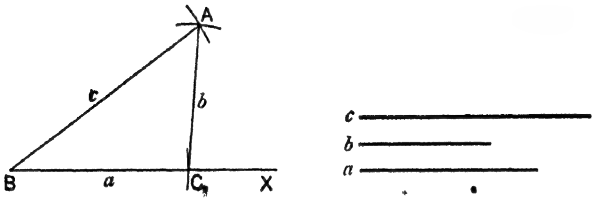
[*Construction.* From P draw PH perp. to AB, and produce PH to P′, making HP′ equal to PH. Join P′Q cutting AB at K. Join PK. Prove that PK, QK are the required lines.]

10. Through a given point P draw a straight line such that the perpendiculars drawn to it from two points A and B may be equal.
Is this always possible?

THE CONSTRUCTION OF TRIANGLES.

PROBLEM 8.

To draw a triangle having given the lengths of the three sides.



Let a, b, c be the lengths to which the sides of the required triangle are to be equal.

Construction. Draw any straight line BX, and cut off from it a part BC equal to a.

With centre B, and radius c, draw an arc of a circle.

With centre C, and radius b, draw a second arc cutting the first at A.

Join AB, AC.

Then ABC is the required triangle, for by construction the sides BC, CA, AB are equal to a, b, c respectively.

Obs. The three data, a, b, c may be understood in two ways: either as three actual lines to which the sides of the triangle are to be equal, or as three *numbers* expressing the lengths of those lines in terms of inches, centimetres, or some other linear unit.

NOTES. (i) In order that the construction may be possible it is necessary that any two of the given sides should be together greater than the third side (Theorem 11); for otherwise the arcs drawn from the centres B and C would not cut.

(ii) The arcs which cut at A would, if continued, cut again on the other side of BC. Thus the construction gives two triangles on opposite sides of a common base.

ON THE CONSTRUCTION OF TRIANGLES.

It has been seen (Page 50) that to prove two triangles identically equal, *three* parts of one must be given equal to the corresponding parts of the other (though *any* three parts do not necessarily serve the purpose). This amounts to saying that *to determine the shape and size of a triangle we must know* **three** *of its parts*: or, in other words,

To construct a triangle three independent data are required.

For example, we may construct a triangle

(i) When *two sides* (b, c) and the *included angle* (A) are given.

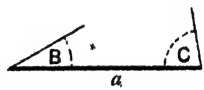
The method of construction in this case is obvious.

(ii) When *two angles* (A, B) and one side (a) are given.

Here, since A and B are given, we at once know C;

for $A+B+C=180°$.

Hence we have only to draw the base equal to a, and at its ends make angles equal to B and C; for we know that the remaining angle must necessarily be equal to A.



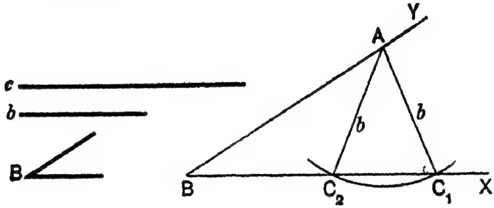
(iii) If the *three angles* A, B, C are given (and no side), the problem is **indeterminate**, that is, the number of solutions is unlimited.

For if at the ends of *any* base we make angles equal to B and C, the third angle is equal to A.

This construction is indeterminate, because the three data are not *independent*, the third following necessarily from the other two.

PROBLEM 9.

To construct a triangle having given two sides and an angle opposite to one of them.



Let b, c be the given sides and B the given angle.

Construction. Take any straight line BX, and at B make the $\angle$XBY equal to the given $\angle$B.

From BY cut off BA equal to c.

With centre A, and radius b, draw an arc of a circle.

If this arc cuts BX in two points C_1 and C_2, both on the same side of B, both the $\triangle$s ABC_1, ABC_2 satisfy the given conditions.

This double solution is known as the **Ambiguous Case**, and will occur when b is less than c but greater than the perp. from A on BX.

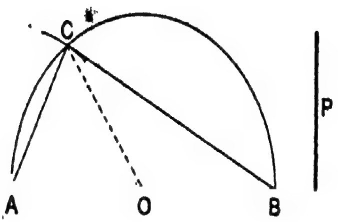
EXERCISE.

Draw figures to illustrate the nature and number of solutions in the following cases:

(i) When b is greater than c.
(ii) When b is equal to c.
(iii) When b is equal to the perpendicular from A on BX.
(iv) When b is less than this perpendicular.

Problem 10.

To construct a right-angled triangle having given the hypotenuse and one side.



Let AB be the hypotenuse and P the given side.

Construction. Bisect AB at O; and with centre O, and radius OA, draw a semicircle.

With centre A, and radius P, draw an arc to cut the semicircle at C.

Join AC, BC.

Then ABC is the required triangle.

Proof. Join OC.

Because OA = OC;

$\therefore$ the $\angle$ OCA = the $\angle$ OAC.

And because OB = OC;

$\therefore$ the $\angle$ OCB = the $\angle$ OBC.

$\therefore$ the whole $\angle$ ACB = the $\angle$ OAC + the $\angle$ OBC

$= \frac{1}{2}$ of $180°$ *Theor.* 16.

$= 90°$.

ON THE CONSTRUCTION OF TRIANGLES.

(Graphical Exercises.)

1. Draw a triangle whose sides are 7·5 cm., 6·2 cm., and 5·3 cm.

Draw and measure the perpendiculars dropped on these sides from the opposite vertices.

[N.B. The perpendiculars, if correctly drawn, will meet at a point, as will be seen later. See page 207.]

2. Draw a triangle ABC, having given $a=3{\cdot}00''$, $b=2{\cdot}50''$, $c=2{\cdot}75''$.

Bisect the angle A by a line which meets the base at X. Measure BX and XC (to the nearest hundredth of an inch); and hence calculate the value of $\frac{BX}{CX}$ to two places of decimals. Compare your result with the value of $\frac{c}{b}$.

3. Two sides of a triangular field are 315 yards and 260 yards, and the included angle is known to be 39°. Draw a plan (1 inch to 100 yards) and find by measurement the length of the remaining side of the field.

4. ABC is a triangular plot of ground, of which the base BC is 75 metres, and the angles at B and C are 47° and 68° respectively. Draw a plan (scale 1 cm. to 10 metres). Write down without measurement the size of the angle A; and by measuring the plan, obtain the approximate lengths of the other sides of the field; also the perpendicular drawn from A to BC.

5. A yacht on leaving harbour steers N.E. sailing 9 knots an hour. After 20 minutes she goes about, steering N.W. for 35 minutes and making the same average speed as before. How far is she now from the harbour, and what course (approximately) must she set for the run home? Obtain your results from a chart of the whole course, scale 2 cm. to 1 knot.

6. Draw a right-angled triangle, given that the hypotenuse $c=10{\cdot}6$ cm. and one side $a=5{\cdot}6$ cm. Measure the third side b; and find the value of $\sqrt{c^2-a^2}$. Compare the two results.

7. Construct a triangle, having given the following parts: $B=34°$, $b=5{\cdot}5$ cm., $c=8{\cdot}5$ cm. Shew that there are two solutions. Measure the two values of a, and also of C, and shew that the latter are supplementary.

8. In a triangle ABC, the angle $A=50°$, and $b=6{\cdot}5$ cm. Illustrate by figures the cases which arise in constructing the triangle, when (i) $a=7$ cm. (ii) $a=6$ cm. (iii) $a=5$ cm. (iv) $a=4$ cm.

9. Two straight roads, which cross at right angles at A, are carried over a straight canal by bridges at B and C. The distance between the bridges is 461 yards, and the distance from the crossing A to the bridge B is 261 yards. Draw a plan, and by measurement of it ascertain the distance from A to C.

(*Problems. State your construction, and give a theoretical proof.*)

10. Draw an isosceles triangle on a base of 4 cm., and having an altitude of 6·2 cm. *Prove* the two sides equal, and measure them to the nearest millimetre.

11. Draw an isosceles triangle having its vertical angle equal to a given angle, and the perpendicular from the vertex on the base equal to a given straight line.

Hence draw an equilateral triangle in which the perpendicular from one vertex on the opposite side is 6 cm. Measure the length of a side to the nearest millimetre.

12. Construct a triangle ABC in which the perpendicular from A on BC is 5·0 cm., and the sides AB, AC are 5·8 cm. and 9·0 cm. respectively. Measure BC.

13. Construct a triangle ABC having the angles at B and C equal to two given angles L and M, and the perpendicular from A on BC equal to a given line P.

14. Construct a triangle ABC (without protractor) having given two angles B and C and the side b.

15. On a given base construct an isosceles triangle having its vertical angle equal to a given angle L.

16. Construct a right-angled triangle, having given the length of the hypotenuse c, and the sum of the remaining sides a and b.

If $c=5{\cdot}3$ cm., and $a+b=7{\cdot}3$ cm., find a and b graphically; and calculate the value of $\sqrt{a^2+b^2}$.

17. Construct a triangle having given the perimeter and the angles at the base. For example, $a+b+c=12$ cm., $B=70°$, $C=80°$.

18. Construct a triangle ABC from the following data:

$$a=6{\cdot}5 \text{ cm.}, \quad b+c=10 \text{ cm.}, \quad \text{and } B=60°.$$

Measure the lengths of b and c.

19. Construct a triangle ABC from the following data:

$$a=7 \text{ cm.}, \quad c-b=1 \text{ cm.}, \quad \text{and } B=55°.$$

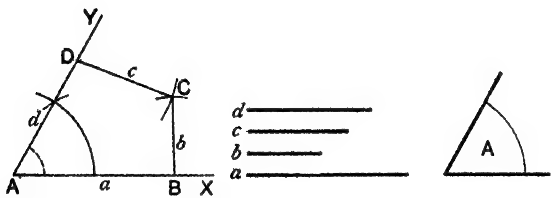
Measure the lengths of b and c.

THE CONSTRUCTION OF QUADRILATERALS.

It has been shewn that the shape and size of a triangle are completely determined when the lengths of its three sides are given. A quadrilateral, however, is not completely determined by the lengths of its four sides. From what follows it will appear that *five* independent data are required to construct a quadrilateral.

PROBLEM 11.

To construct a quadrilateral, given the lengths of the four sides, and one angle.



Let a, b, c, d be the given lengths of the sides, and A the angle between the sides equal to a and d.

Construction. Take any straight line AX, and cut off from it AB equal to a.

Make the $\angle$ BAY equal to the $\angle$ A.

From AY cut off AD equal to d.

With centre D, and radius c, draw an arc of a circle.

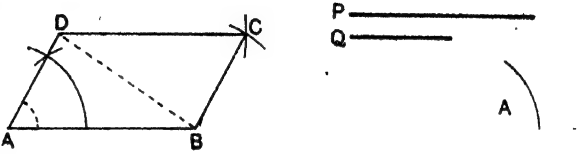
With centre B and radius b, draw another arc to cut the former at C.

Join DC, BC.

Then ABCD is the required quadrilateral; for by construction the sides are equal to a, b, c, d, and the $\angle$ DAB is equal to the given angle.

PROBLEM 12.

To construct a parallelogram having given two adjacent sides and the included angle.



Let P and Q be the two given sides, and A the given angle.

Construction 1. (*With ruler and compasses.*) Take a line AB equal to P; and at A make the $\angle$ BAD equal to the $\angle$ A, and make AD equal to Q.

With centre D, and radius P, draw an arc of a circle.

With centre B, and radius Q, draw another arc to cut the former at C.

Then ABCD is the required par^m.

Proof. Join DB.

In the $\triangle^s$ DCB, BAD,

because $\begin{cases} DC = BA, \\ CB = AD, \end{cases}$
and DB is common;

$\therefore$ the $\angle$ CDB = the $\angle$ ABD; *Theor.* 7.

and these are alternate angles,

$\therefore$ DC is par^l to AB.

Also DC = AB;

$\therefore$ DA and BC are also equal and parallel. *Theor.* 20.

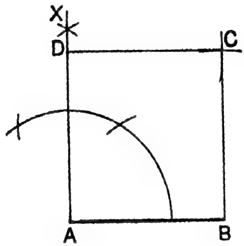
$\therefore$ ABCD is a par^m.

Construction 2. (*With set squares.*) Draw AB and AD as efore; then with set squares through D draw DC par^l to AB, nd through B draw BC par^l to AD.

By construction, ABCD is a par^m having the required parts.

PROBLEM 13.

To construct a square on a given side.



Let AB be the given side.

Construction 1. (*With ruler and compasses.*) At A draw AX perp. to AB, and cut off from it AD equal to AB.

With B and D as centres, and with radius AB, draw two arcs cutting at C.

Join BC, DC.

Then ABCD is the required square.

Proof. As in Problem 12, ABCD may be shewn to be a par^{m}. And since the $\angle$BAD is a right angle, the figure is a rectangle. Also, by construction all its sides are equal.

$\therefore$ ABCD is a square.

Construction 2. (*With set squares.*) At A draw AX perp. to AB, and cut off from it AD equal to AB.

Through D draw DC par^{l} to AB, and through B draw BC par^{l} to AD meeting DC in C.

Then, by construction, ABCD is a rectangle. [Def. 3, page 56.]

Also it has the two adjacent sides AB, AD equal.

$\therefore$ it is a square.

EXERCISES.

ON THE CONSTRUCTION OF QUADRILATERALS.

1. Draw a rhombus each of whose sides is equal to a given straight line PQ, which is also to be one diagonal of the figure.

Ascertain (without measurement) the number of degrees in each angle, giving a reason for your answer.

2. Draw a square on a side of 2·5 inches. Prove theoretically that its diagonals are equal; and by measuring the diagonals to the nearest hundredth of an inch test the correctness of your drawing.

3. Construct a square on a diagonal of 3·0″, and measure the lengths of each side. Obtain the average of your results.

4. Draw a parallelogram ABCD, having given that one side AB = 5·5 cm., and the diagonals AC, BD are 8 cm., and 6 cm. respectively. Measure AD.

5. The diagonals of a certain quadrilateral are equal, (each 6·0 cm.), and they bisect one another at an angle of 60°. Shew that *five* independent data are here given.

Construct the quadrilateral. Name its species; and give a *formal* proof of your answer. Measure the perimeter. If the angle between the diagonals were increased to 90°, by how much per cent. would the perimeter be increased?

6. In a quadrilateral ABCD,

AB = 5·6 cm., BC = 2·5 cm., CD = 4·0 cm., and DA = 3·3 cm.

Shew that the shape of the quadrilateral is not settled by these data.

Draw the quadrilateral when (i) A = 30° (ii) A = 60°. Why does the construction fail when A = 100°?

Determine graphically the least value of A for which the construction fails.

7. Shew how to construct a quadrilateral, having given the lengths of the four sides and of one diagonal. What conditions must hold among the data in order that the problem may be possible?

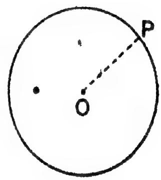
Illustrate your method by constructing a quadrilateral ABCD, when

(i) AB = 3·0″, BC = 1·7″, CD = 2·5″, DA = 2·8″, and the diagonal BD = 2·6″. Measure AC.

(ii) AB = 3·6 cm., BC = 7·7 cm., CD = 6·8 cm., DA = 5·1 cm., and the diagonal AC = 8·5 cm. Measure the angles at B and D.

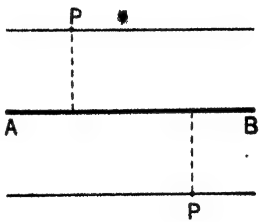
LOCI.

DEFINITION. The **locus** of a point is the path traced out by it when it moves in accordance with some given law.



Example 1. Suppose the point P to move so that its distance from a fixed point O is **constant** (say 1 centimetre).

Then the locus of P is evidently the circumference of a circle whose centre is O and radius 1 cm.



Example 2. Suppose the point P moves at a constant distance (say 1 cm.) from a fixed straight line AB.

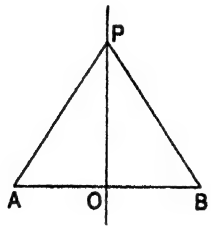
Then the locus of P is one or other of two straight lines parallel to AB, on either side, and at a distance of 1 cm. from it.

Thus the locus of a point, moving under some given condition, consists of the line or lines to which the point is thereby restricted; provided that the condition is satisfied by every point on such line or lines, and by no other.

When we find a series of points which satisfy the given law, and through which therefore the moving point must pass, we are said to **plot the locus** of the point.

PROBLEM 14.

To find the locus of a point P which moves so that its distances from two fixed points A and B are always equal to one another.



Here the point P moves through all positions in which PA = PB;
∴ one position of the moving point is at O the middle point of AB.

Suppose P to be *any* other position of the moving point; that is, let PA = PB.

Join OP.

Then in the △ˢ POA, POB,

because { PO is common, OA = OB, and PA = PB, by hypothesis;

∴ the ∠ POA = the ∠ POB. *Theor.* 7.

Hence PO is perpendicular to AB.

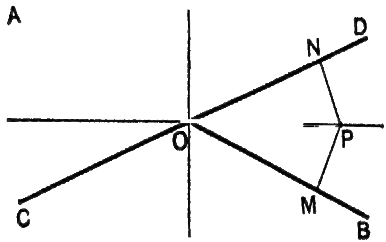
That is, *every point* P *which is equidistant from* A *and* B *lies on the straight line bisecting* AB *at right angles.*

Likewise it may be proved that every point on the perpendicular through O is equidistant from A and B.

This line is therefore the required locus.

PROBLEM 15.

To find the locus of a point P *which moves so that its perpendicular distances from two given straight lines* AB, CD *are equal to one another.*



Let P be any point such that the perp. PM = the perp. PN.

Join P to O, the intersection of AB, CD.

Then in the $\triangle^s$ PMO, PNO,

because $\begin{cases} \text{the } \angle^s \text{ PMO, PNO are } \textit{right angles}, \\ \text{the hypotenuse OP is common,} \\ \text{and one side PM} = \text{one side PN;} \end{cases}$

$\therefore$ the triangles are equal in all respects; *Theor.* 18.
so that the $\angle$ POM = the $\angle$ PON.

Hence, if P lies within the $\angle$ BOD, it must be on the bisector of that angle;

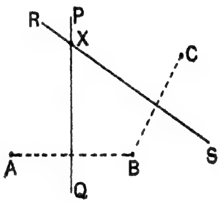
and, if P is within the $\angle$ AOD, it must be on the bisector of that angle.

It follows that *the required locus is the pair of lines which bisect the angles between* AB *and* CD.

INTERSECTION OF LOCI.

The method of Loci may be used to find the position of a point which is subject to two conditions. For corresponding to each condition there will be a locus on which the required point must lie. Hence all points which are common to these two loci, that is, all the points of intersection of the loci, will satisfy *both* the given conditions.

EXAMPLE 1. *To find a point equidistant from three given points* A, B, C, *which are not in the same straight line.*

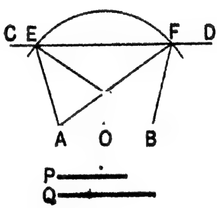


(i) The locus of points equidistant from A and B is the straight line PQ, which bisects AB at right angles.

(ii) Similarly, the locus of points equidistant from B and C is the straight line RS which bisects BC at right angles.

Hence the point common to PQ and RS must satisfy both conditions: that is to say, X the point of intersection of PQ and RS will be equidistant from A, B, and C.

EXAMPLE 2. *To construct a triangle, having given the base, the altitude, and the length of the median which bisects the base.*



Let AB be the given base, and P and Q the lengths of the altitude and median respectively.

Then the triangle is known if its *vertex* is known.

(i) Draw a straight line CD parallel to AB, and at a distance from it equal to P: *then the required vertex must lie on* CD.

(ii) Again, from O the middle point of AB as centre, with radius equal to Q, describe a circle:

then the required vertex must lie on this circle.

Hence any points which are common to CD and the circle, satisfy both the given conditions: that is to say, if CD intersect the circle in E, F, each of the points of intersection might be the vertex of the required triangle. This supposes the length of the median Q to be greater than the altitude.

It may happen that the data of the problem are so related to one another that the resulting loci do not intersect. In this case the problem is impossible.

Obs. In examples on the Intersection of Loci the student should make a point of investigating the relations which must exist among the data, in order that the problem may be possible; and he must observe that if under certain relations *two* solutions are possible, and under other relations no solution exists, there will always be some *intermediate* relation under which the two solutions combine in a single solution.

EXAMPLES ON LOCI.

1. Find the locus of a point which moves so that its distance (measured radially) from the circumference of a given circle is constant.

2. A point P moves along a straight line RQ; find the position in which it is equidistant from two given points A and B.

3. A and B are two fixed points within a circle: find points on the circumference equidistant from A and B. How many such points are there?

4. A point P moves along a straight line RQ; find the position in which it is equidistant from two given straight lines AB and CD.

5. A and B are two fixed points 6 cm. apart. Find by the method of loci two points which are 4 cm. distant from A, and 5 cm. from B.

6 AB and CD are two given straight lines. Find points 3 cm. distant from AB, and 4 cm. from CD. How many solutions are there?

7. A straight rod of given length slides between two straight rulers placed at right angles to one another.

Plot the locus of its middle point; and shew that this locus is the fourth part of the circumference of a circle. [See Problem 10.]

8. On a given base as hypotenuse right-angled triangles are described. Find the locus of their vertices.

9. A is a fixed point, and the point X moves on a fixed straight line BC.

Plot the locus of P, the middle point of AX; and *prove* the locus to be a straight line parallel to BC.

10. A is a fixed point, and the point X moves on the circumference of a given circle.

Plot the locus of P, the middle point of AX; and *prove* that this locus is a circle. [See Ex. 3, p. 64.]

11. AB is a given straight line, and AX is the perpendicular drawn from A to any straight line passing through B. If BX revolve about B, find the locus of the middle point of AX.

12. Two straight lines OX, OY cut at right angles, and from P, a point within the angle XOY, perpendiculars PM, PN are drawn to OX, OY respectively. Plot the locus of P when

(i) $PM + PN$ is constant ($=6$ cm., say);

(ii) $PM - PN$ is constant ($=3$ cm., say).

And in each case give a theoretical proof of the result you arrive at experimentally.

13. Two straight lines OX, OY intersect at right angles at O; and from a movable point P perpendiculars PM, PN are drawn to OX, OY.

Plot (without proof) the locus of P, when

(i) $PM = 2\ PN$;

(ii) $PM = 3\ PN$.

14. Find a point which is at a given distance from a given point, and is equidistant from two given parallel straight lines.

When does this problem admit of two solutions, when of one only, and when is it impossible?

15. S is a fixed point 2 inches distant from a given straight line MX. Find two points which are $2\frac{3}{4}$ inches distant from S, and also $2\frac{3}{4}$ inches distant from MX.

16. Find a series of points equidistant from a given point S and a given straight line MX. Draw a curve freehand passing through all the points so found.

17. On a given base construct a triangle of given altitude, having its vertex on a given straight line.

18. Find a point equidistant from the three sides of a triangle.

19. Two straight lines OX, OY cut at right angles; and Q and R are points in OX and OY respectively. Plot the locus of the middle point of QR, when

(i) $OQ + OR =$ constant.

(ii) $OQ - OR =$ constant.

20. S and S′ are two fixed points. Find a series of points P such that

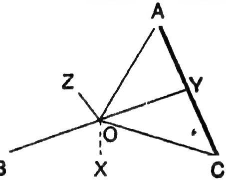
(i) $SP + S'P =$ constant (say 3·5 inches).

(ii) $SP - S'P =$ constant (say 1·5 inch).

In each case draw a curve freehand passing through all the points so found.

ON THE CONCURRENCE OF STRAIGHT LINES IN A TRIANGLE.

I. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.*



Let ABC be a △, and X, Y, Z the middle points of its sides.

From Z and Y draw perps. to AB, AC, meeting at O. *Join* OX.

It is required to prove that OX *is perp. to* BC.

Join OA, OB, OC.

Proof. Because YO bisects AC at right angles,
∴ it is the locus of points equidistant from A and C;
∴ OA = OC.

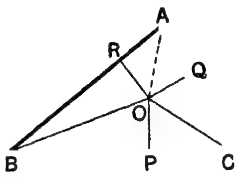
Again, because ZO bisects AB at right angles,
∴ it is the locus of points equidistant from A and B;
∴ OA = OB.

Hence OB = OC.
∴ O is on the locus of points equidistant from B and C:
that is, OX is perp. to BC.

Hence the perpendiculars from the mid-points of the sides meet at O.

Q.E.D.

II. *The bisectors of the angles of a triangle are concurrent.*



Let ABC be a △. Bisect the ∠s ABC, BCA by straight lines which meet at O.

Join AO.

It is required to prove that AO *bisects the* ∠ BAC.

From O draw OP, OQ, OR perp. to the sides of the △.

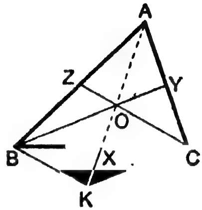
Proof. Because BO bisects the ∠ ABC,
∴ it is the locus of points equidistant from BA and BC;
∴ OP = OR.

Similarly CO is the locus of points equidistant from BC and CA;
∴ OP = OQ.

Hence OR = OQ.
∴ O is on the locus of points equidistant from AB and AC;
that is, OA is the bisector of the ∠ BAC.

Hence the bisectors of the angles meet at O. Q.E.D.

III. *The medians of a triangle are concurrent.*



Let ABC be a $\triangle$.
Let BY and CZ be two of its medians, and let them intersect at O.

Join AO,
and produce it to meet BC in X.

It is required to shew that AX *is the remaining median of the* $\triangle$.

Through C draw CK parallel to BY;
produce AX to meet CK at K.
Join BK.

Proof. In the $\triangle$ AKC,
because Y is the middle point of AC, and YO is parallel to CK,
$\therefore$ O is the middle point of AK. *Theor.* 22.

Again in the $\triangle$ ABK,
since Z and O are the middle points of AB, AK,
$\therefore$ ZO is parallel to BK,
that is, OC is parallel to BK.
$\therefore$ the figure BKCO is a par^{m}.

But the diagonals of a par^{m} bisect one another;
$\therefore$ X is the middle point of BC.
That is, AX is a median of the $\triangle$.

Hence the three medians meet at the point O. Q.E.D.

DEFINITION. The point of intersection of the medians is called the **centroid** of the triangle.

COROLLARY. *The three medians of a triangle cut one another at a point of trisection, the greater segment in each being towards the angular point.*

For in the above figure it has been proved that
AO = OK,
also that OX is half of OK;
$\therefore$ OX is half of OA:
that is, OX is one third of AX.
Similarly OY is one third of BY,
and OZ is one third of CZ. Q.E.D.

By means of this Corollary it may be shewn that in any triangle the shorter median bisects the greater side.

NOTE. It will be proved hereafter that *the perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*

MISCELLANEOUS PROBLEMS.

(A theoretical proof is to be given in each case.)

1. A is a given point, and BC a given straight line. From A draw a straight line to make with BC an angle equal to a given angle X.

How many such lines can be drawn?

2. Draw the bisector of an angle AOB, without using the vertex O in your construction.

3. P is a given point within the angle AOB. Draw through P a straight line terminated by OA and OB, and bisected at P.

4. OA, OB, OC are three straight lines meeting at O. Draw a transversal terminated by OA and OC, and bisected by OB.

5. Through a given point A draw a straight line so that the par intercepted between two given parallels may be of given length.

When does this problem admit of two solutions? When of only one? And when is it impossible?

6. In a triangle ABC inscribe a rhombus having one of its angles coinciding with the angle A.

7. Use the properties of an equilateral triangle to trisect a given straight line.

(Construction of Triangles.)

8. Construct a triangle, having given

(i) The middle points of the three sides.

(ii) The lengths of two sides and of the median which bisects the third side.

(iii) The lengths of one side and the medians which bisect the other two sides.

(iv) The lengths of the three medians.

PART II.

ON AREAS.

DEFINITIONS.

1. The **altitude** (or **height**) of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side.

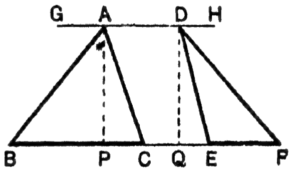
2. The **altitude** (or **height**) of a triangle with reference to a given side as base, is the perpendicular distance of the opposite vertex from the base.

NOTE. It is clear that *parallelograms or triangles which are between the same parallels have the same altitude.*

For let AP and DQ be the altitudes of the $\triangle^s$ ABC, DEF, which are between the same parallels BF, GH.

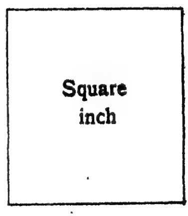
Then the fig. APQD is evidently a rectangle;

$\therefore$ AP = DQ.



3. The **area** of a figure is the amount of surface contained within its bounding lines.

4. A **square inch** is the area of a square drawn on a side one inch in length.



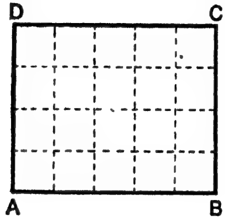
5. Similarly a **square centimetre** is the area of a square drawn on a side one centimetre in length.

The terms *square yard, square foot, square metre* are to be understood in the same sense.

6. Thus the **unit of area** is the area of a square on a side of unit length.

Theorem 23.

Area of a rectangle. *If the number of units in the length of a rectangle is multiplied by the number of units in its breadth, the product gives the number of square units in the area.*



Let ABCD represent a rectangle whose length AB is 5 feet, and whose breadth AD is 4 feet.

Divide AB into 5 equal parts, and BC into 4 equal parts, and through the points of division of each line draw parallels to the other.

The rectangle ABCD is now divided into compartments, each of which represents one square foot.

Now there are 4 rows, each containing 5 squares,
$\therefore$ the rectangle contains 5×4 square feet.

Similarly, if the length $= a$ linear units, and the breadth $= b$ linear units

the rectangle contains **ab** *units of* **area.**

And if each side of a square $= a$ linear units,

the square contains $\mathbf{a^2}$ *units of* **area.**

These statements may be thus abridged:

the area of a rectangle = length × breadth............(i),

the area of a square $= (side)^2$ ....(ii).

Q.E.D.

Corollaries. (i) *Rectangles which have equal lengths and equal breadths have equal areas.*

(ii) *Rectangles which have equal areas and equal lengths have also equal breadths.*

NOTATION.

The rectangle ABCD is said to be **contained** by AB, AD; for these adjacent sides fix its size and shape.

A rectangle whose adjacent sides are AB, AD is denoted by *rect.* AB, AD, or simply $AB \times AD$.

A square drawn on the side AB is denoted by *sq. on* AB, or AB^2.

EXERCISES.

(*On Tables of Length and Area.*)

1. Draw a figure to shew *why*

(i) 1 sq. yard $= 3^2$ sq. feet.
(ii) 1 sq. foot $= 12^2$ sq. inches.
(iii) 1 sq. cm. $= 10^2$ sq. mm.

2. Draw a figure to shew that the square on a straight line is four times the square on half the line.

3. Use squared paper to shew that the square on $1'' = 10^2$ times the square on $0{\cdot}1''$.

4. If $1''$ represents 5 miles, what does an area of 6 square inches represent?

EXTENSION OF THEOREM 23.

The proof of Theorem 23 here given supposes that the length and breadth of the given rectangle are expressed by *whole numbers*; but the formula holds good when the length and breadth are fractional.

This may be illustrated thus:

Suppose the length and breadth are 3·2 cm. and 2·4 cm.; we shall shew that the area is $(3{\cdot}2 \times 2{\cdot}4)$ sq. cm.

For
$$\text{length} = 3{\cdot}2 \text{ cm.} = 32 \text{ mm.}$$
$$\text{breadth} = 2{\cdot}4 \text{ cm.} = 24 \text{ mm.}$$
$$\therefore \text{ area} = (32 \times 24) \text{ sq. mm.} = \frac{32 \times 24}{10^2} \text{ sq. cm.}$$
$$= (3{\cdot}2 \times 2{\cdot}4) \text{ sq. cm.}$$

EXERCISES.

(On the Area of a Rectangle.)

*Draw on squared paper the rectangles of which the length (a) and breadth (b) are given below. Calculate the areas, and verify by the actual counting of squares.

1. $a=2''$, $b=3''$.
2. $a=1{\cdot}5''$, $b=4''$.
3. $a=0{\cdot}8''$, $b=3{\cdot}5''$.
4. $a=2{\cdot}5''$, $b=1{\cdot}4''$.
5. $a=2{\cdot}2''$, $b=1{\cdot}5''$.
6. $a=1{\cdot}6''$, $b=2{\cdot}1''$.

Calculate the areas of the rectangles in which

7. $a=18$ metres, $b=11$ metres.
8. $a=7$ ft., $b=72$ in.
9. $a=2{\cdot}5$ km., $b=4$ metres.
10. $a=\frac{1}{4}$ mile, $b=1$ inch.

11. The area of a rectangle is 30 sq. cm., and its length is 6 cm. Find the breadth. Draw the rectangle on squared paper; and verify your work by counting the squares.

12. Find the length of a rectangle whose area is 3·9 sq. in., and breadth 1·5″. Draw the rectangle on squared paper; and verify your work by counting the squares.

13. (i) When you treble the length of a rectangle without altering its breadth, how many times do you multiply the area?

(ii) When you treble both length and breadth, how many times do you multiply the area?

Draw a figure to illustrate your answers; and state a general rule.

14. In a plan of a rectangular garden the length and breadth are 3·6″ and 2·5″, one inch standing for 10 yards. Find the area of the garden.

If the area is increased by 300 sq. yds., the breadth remaining the same, what will the new length be? And how many inches will represent it on your plan?

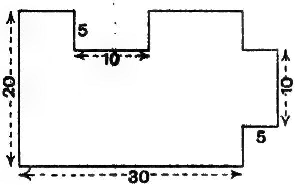
15. Find the area of a rectangular enclosure of which a plan (scale 1 cm. to 20 metres) measures 6·5 cm. by 4·5 cm.

16. The area of a rectangle is 1440 sq. yds. If in a plan the sides of the rectangle are 3·2 cm. and 4·5 cm., on what scale is the plan drawn?

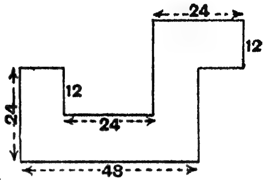
17. The area of a rectangular field is 52000 sq. ft. On a plan of this, drawn to the scale of 1″ to 100 ft., the length is 3·25″. What is the breadth?

Calculate the areas of the enclosures of which plans are given below. All the angles are right angles, and the dimensions are marked in feet.

18.

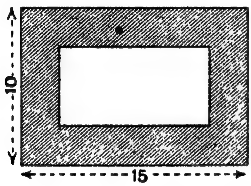


19.



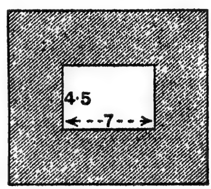
Calculate the areas represented by the shaded parts of the following plans. The dimensions are marked in feet.

20.



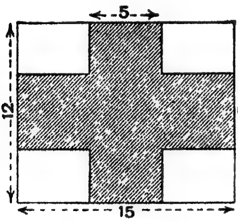
Width of shaded border uniform $2\frac{1}{2}$ ft.

21.

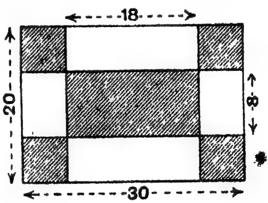


Width of shaded border uniform 4 ft.

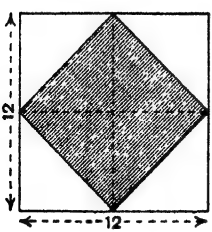
22.



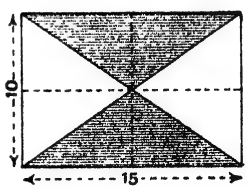
23.



24.

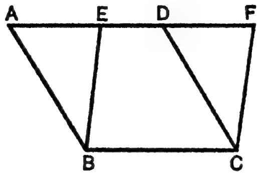


25.



THEOREM 24. [Euclid I. 35.]

Parallelograms on the same base and between the same parallels are equal in area.



Let the par^ms ABCD, EBCF be on the same base BC, and between the same par^ls BC, AF.

It is required to prove that

the par^m ABCD = *the par^m* EBCF *in area.*

Proof. In the $\triangle^s$ FDC, EAB,

because $\begin{cases} \text{DC} = \text{the opp. side AB}; & \textit{Theor. } 21. \\ \text{the ext. } \angle \text{FDC} = \text{the int. opp. } \angle \text{EAB}; & \textit{Theor. } 14. \\ \text{the int. } \angle \text{DFC} = \text{the ext. } \angle \text{AEB}; & \end{cases}$

$\therefore$ the $\triangle$FDC = the $\triangle$EAB. *Theor.* 17.

Now, if from the whole fig. ABCF the $\triangle$FDC is taken, the remainder is the par^m ABCD.

And if from the whole fig. ABCF the $\triangle$EAB is taken, the remainder is the par^m EBCF.

$\therefore$ these remainders are equal;

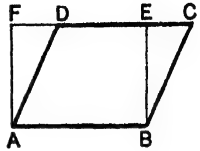
that is, the par^m ABCD = the par^m EBCF. Q.E.D.

EXERCISE.

In the above diagram the sides AD, EF overlap. Draw diagrams in which (i) these sides do not overlap; (ii) the ends E and D coincide.

Go through the proof with these diagrams, and ascertain if it applies to them without change.

THE AREA OF A PARALLELOGRAM.



Let ABCD be a parallelogram, and ABEF the rectangle on the same base AB and of the same altitude BE. Then by Theorem 24,

area of par^{m} ABCD = area of rect. ABEF
= AB × BE
= *base × altitude.*

COROLLARY. Since the area of a parallelogram depends only on its base and altitude, it follows that

Parallelograms on equal bases and of equal altitudes are equal in area.

EXERCISES.

(Numerical and Graphical.)

1. Find the area of parallelograms in which
   (i) the base = 5·5 cm., and the height = 4 cm.
   (ii) the base = 2·4″, and the height = 1·5″.

2. Draw a parallelogram ABCD having given AB = $2\frac{1}{2}$″, AD = $1\frac{1}{2}$″, and the ∠A = 65°. Draw and measure the perpendicular from D on AB, and hence calculate the approximate area. Why *approximate*?

Again calculate the area from the length of AD and the perpendicular on it from B. Obtain the average of the two results.

3. Two adjacent sides of a parallelogram are 30 metres and 25 metres, and the included angle is 50°. Draw a plan, 1 cm. representing 5 metres; and by measuring each altitude, make two independent calculations of the area. Give the average result.

4. The area of a parallelogram ABCD is 4·2 sq. in., and the base AB is 2·8″. Find the height. If AD = 2″, draw the parallelogram.

5. Each side of a rhombus is 2″, and its area is 3·86 sq. in. Calculate an altitude. Hence draw the rhombus, and measure one of its acute angles.

THEOREM 25.

The Area of a Triangle. *The area of a triangle is half the area of the rectangle on the same base and having the same altitude.*

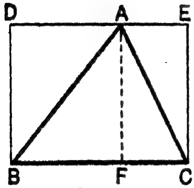


Fig. 1.

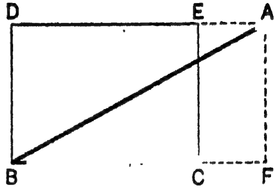


Fig. 2.

Let ABC be a triangle, and BDEC a rectangle on the same base BC and with the same altitude AF.

It is required to prove that the $\triangle$ ABC *is half the rectangle* BDEC.

Proof. Since AF is perp. to BC, each of the figures DF, EF is a rectangle.

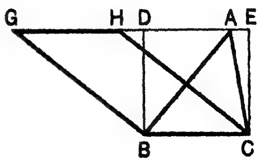
Because the diagonal AB bisects the rectangle DF,
$\therefore$ the $\triangle$ ABF is half the rectangle DF.

Similarly, the $\triangle$ AFC is half the rectangle FE.

$\therefore$ adding these results in Fig. 1, and taking the difference in Fig. 2,
the $\triangle$ ABC is half the rectangle BDEC.

Q.E.D.

COROLLARY. *A triangle is half any parallelogram on the same base and between the same parallels.*



For the $\triangle$ ABC is half the rect. BCED.

And the rect. BCED = any par$^{\text{m}}$ BCHG on the same base and between the same par$^{\text{ls}}$.

$\therefore$ the $\triangle$ ABC is half the par$^{\text{m}}$ BCHG.

THE AREA OF A TRIANGLE.

If BC and AF respectively contain a units and p units of length, the rectangle BDEC contains ap units of area.

$\therefore$ the area of the $\triangle$ ABC $= \frac{1}{2}ap$ units of area.

This result may be stated thus:

Area of a Triangle $= \frac{1}{2}$. *base* $\times$ *altitude.*

EXERCISES ON THE AREA OF A TRIANGLE.

(*Numerical and Graphical.*)

Calculate the areas of the triangles in which

(i) the base $=24$ ft., the height $=15$ ft.

(ii) the base $=4{\cdot}8''$, the height $=3{\cdot}5''$.

(iii) the base $=160$ metres, the height $=125$ metres.

2. Draw triangles from the following data. In each case draw and measure the altitude with reference to a given side as base: hence calculate the approximate area.

(i) $a=8{\cdot}4$ cm., $b=6{\cdot}8$ cm., $c=4{\cdot}0$ cm.

(ii) $b=5{\cdot}0$ cm., $c=6{\cdot}8$ cm., A$=65°$.

(iii) $a=6{\cdot}5$ cm., B$=52°$, C$=76°$.

3. ABC is a triangle *right-angled* at C; shew that its area $=\frac{1}{2}$ BC $\times$ CA.

Given $a=6$ cm., $b=5$ cm., calculate the area.

Draw the triangle and measure the hypotenuse c; draw and measure the perpendicular from C on the hypotenuse; hence calculate the approximate area.

Note the error in your approximate result, and express it as a percentage of the true value.

4. Repeat the whole process of the last question for a right-angled triangle ABC, in which $a=2{\cdot}8''$ and $b=4{\cdot}5''$; C being the right angle as before.

5. In a triangle, given

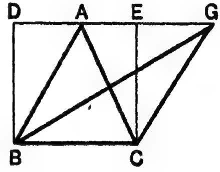
(i) Area $=80$ sq. in., base $=1$ ft. 8 in.; calculate the altitude.

(ii) Area $=10{\cdot}4$ sq. cm., altitude $=1{\cdot}6$ cm.; calculate the base.

6. Construct a triangle ABC, having given $a=3{\cdot}0''$, $b=2{\cdot}8''$, $c=2{\cdot}6''$. Draw and measure the perpendicular from A on BC; hence calculate the approximate area.

THEOREM 26. [Euclid I. 37.]

Triangles on the same base and between the same parallels (hence, of the same altitude) are equal in area.



Let the $\triangle^s$ ABC, GBC be on the same base BC and between the same par^{ls} BC, AG.

It is required to prove that the $\triangle$ ABC = *the* $\triangle$ GBC *in area.*

Proof. If BCED is the rectangle on the base BC, and between the same parallels as the given triangles,

the $\triangle$ ABC is half the rect. BCED; *Theor.* 25.

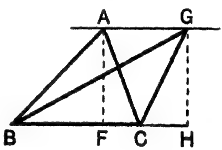
also the $\triangle$ GBC is half the rect. BCED;

$\therefore$ the $\triangle$ ABC = the $\triangle$ GBC. Q.E.D.

Similarly, *triangles on equal bases and of equal altitudes are equal in area.*

THEOREM 27. [Euclid I. 39.]

If two triangles are equal in area, and stand on the same base and on the same side of it, they are between the same parallels.



Let the $\triangle^s$ ABC, GBC, standing on the same base BC, be equal in area; and let AF and GH be their altitudes.

It is required to prove that AG *and* BC *are par^l.*

Proof. The $\triangle$ ABC is half the rectangle contained by BC and AF;

and the $\triangle$ GBC is half the rectangle contained by BC and GH;

$\therefore$ the rect. BC, AF = the rect. BC, GH;

$\therefore$ AF = GH. *Theor.* 23, *Cor.* 2.

Also AF and GH are par^l;

hence AG and FH, that is BC, are par^l. Q.E.D.

EXERCISES ON THE AREA OF A TRIANGLE.

(Theoretical.)

1. ABC is a triangle and XY is drawn parallel to the base BC, cutting the other sides at X and Y. Join BY and CX; and shew that

(i) the $\triangle$ XBC = the $\triangle$ YBC;
(ii) the $\triangle$ BXY = the $\triangle$ CXY;
(iii) the $\triangle$ ABY = the $\triangle$ ACX.

If BY and CX cut at K, shew that

(iv) the $\triangle$ BKX = the $\triangle$ CKY.

2. Shew that a median of a triangle divides it into two parts of equal area.

How would you divide a triangle into *three* equal parts by straight lines drawn from its vertex?

3. Prove that a parallelogram is divided by its diagonals into four triangles of equal area.

4. ABC is a triangle whose base BC is bisected at X. If Y is any point in the median AX, shew that

the $\triangle$ ABY = the $\triangle$ ACY in area.

5. ABCD is a parallelogram, and BP, DQ are the perpendiculars from B and D on the diagonal AC.

Shew that BP = DQ.

Hence if X is any point in AC, or AC produced,

prove (i) the $\triangle$ ADX = the $\triangle$ ABX;
(ii) the $\triangle$ CDX = the $\triangle$ CBX.

6. Prove by means of Theorems 26 and 27 that *the straight line joining the middle points of two sides of a triangle is parallel to the third side.*

7. The straight line which joins the middle points of the oblique sides of a trapezium is parallel to each of the parallel sides.

8. ABCD is a parallelogram, and X, Y are the middle points of the sides AD, BC; if Z is any point in XY, or XY produced, shew that the triangle AZB is one quarter of the parallelogram ABCD.

9. If ABCD is a parallelogram, and X, Y any points in DC and AD respectively: shew that the triangles AXB, BYC are equal in area.

10. ABCD is a parallelogram, and P is any point within it; shew that the sum of the triangles PAB, PCD is equal to half the parallelogram.

EXERCISES ON THE AREA OF A TRIANGLE.

(*Numerical and Graphical.*)

1. The sides of a triangular field are 370 yds., 200 yds., and 190 yds. Draw a plan (scale 1″ to 100 yards). Draw and measure an altitude; hence calculate the approximate area of the field in square yards.

2. Two sides of a triangular enclosure are 124 metres and 144 metres respectively, and the included angle is observed to be 45°. Draw a plan (scale 1 cm. to 20 metres). Make any necessary measurement, and calculate the approximate area.

3. In a triangle ABC, given that the area $=6{\cdot}6$ sq. cm., and the base $BC=5{\cdot}5$ cm., find the altitude. Hence determine the locus of the vertex A.

If in addition to the above data, $BA=2{\cdot}6$ cm., construct the triangle; and measure CA.

4. In a triangle ABC, given *area* $=3{\cdot}06$ sq. in., and $a=3{\cdot}0''$. Find the altitude, and the locus of A. Given $C=68°$, construct the triangle; and measure b.

5. ABC is a triangle in which BC, BA have constant lengths 6 cm. and 5 cm. If BC is fixed, and BA revolves about B, trace the changes in the area of the triangle as the angle B increases from 0° to 180°.

Answer this question by drawing a series of triangles, increasing B by increments of 30°. Find the area in each case and tabulate the results.

(*Theoretical.*)

6. If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides *supplementary*, shew that the triangles are equal *in area*. Can such triangles ever be *identically* equal?

7. Shew how to draw on the base of a given triangle an isosceles triangle of equal area.

8. If the middle points of the sides of a quadrilateral are joined in order, prove that the *parallelogram* so formed [see Ex. 7, p. 64], is half the quadrilateral.

9. ABC is a triangle, and R, Q the middle points of the sides AB, AC; shew that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral AQXR.

10. Two triangles of equal area stand on the same base but on opposite sides of it: shew that the straight line joining their vertices is bisected by the base, or by the base produced.

[The method given below may be omitted from a first course. In any case it must be postponed till Theorem 29 has been read.]

The Area of a Triangle. *Given the three sides of a triangle, to calculate the area.*

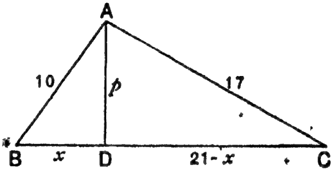
EXAMPLE. *Find the area of a triangle whose sides measure* 21 m., 17 m., *and* 10 m.

Let ABC represent the given triangle.

Draw AD perp. to BC, and denote AD by p.

We shall first find the length of BD.

Let BD $=x$ metres; then DC $=21-x$ metres.



From the right-angled △ ADB, we have by Theorem 29

$$AD^2=AB^2-BD^2=10^2-x^2.$$

And from the right-angled △ ADC,

$$AD^2=AC^2-DC^2=17^2-(21-x)^2;$$

$$\therefore\ 10^2-x^2=17^2-(21-x)^2$$

or $$100-x^2=289-441+42x-x^2;$$

whence $$x=6.$$

Again, $$AD^2=AB^2-BD^2;$$

or $$p^2=10^2-6^2=64;$$

$$\therefore\ p=8.$$

Now *Area of triangle* $=\frac{1}{2}$. *base* × *altitude*

$$=(\tfrac{1}{2}\times 21\times 8)\text{ sq. m.}=84\text{ sq. m.}$$

EXERCISES.

Find by the above method the area of the triangles, whose sides are as follows:

1. 20 ft., 13 ft., 11 ft.
2. 15 yds., 14 yds., 13 yds.
3. 21 m., 20 m., 13 m.
4. 30 cm., 25 cm., 11 cm.
5. 37 ft., 30 ft., 13 ft.
6. 51 m., 37 m., 20 m.
7. If the given sides are a, b and c units in length, prove

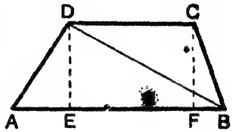
(i) $x=\dfrac{a^2+c^2-b^2}{2a}$; (ii) $p^2=c^2-\left\{\dfrac{a^2+c^2-b^2}{2a}\right\}^2$;

(iii) $\Delta=\frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$.

THE AREA OF QUADRILATERALS.

THEOREM 28.

To find the area of (i) *a trapezium.*
(ii) *any quadrilateral.*



(i) Let ABCD be a trapezium, having the sides AB, CD parallel. Join BD, and from C and D draw perpendiculars CF, DE to AB.

Let the parallel sides AB, CD measure a and b units of length, and let the height CF contain h units.

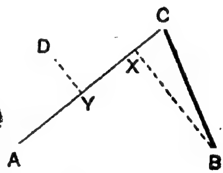
Then the area of ABCD $= \triangle ABD + \triangle DBC$

$$= \frac{1}{2} AB \cdot DE + \frac{1}{2} DC \cdot CF$$

$$= \frac{1}{2} ah + \frac{1}{2} bh = \frac{h}{2}(a+b).$$

That is,

the area of a trapezium $= \frac{1}{2}$ *height* $\times$ (*the sum of the parallel sides*).



(ii) Let ABCD be any quadrilateral. Draw a diagonal AC; and from B and D draw perpendiculars BX, DY to AC. These perpendiculars are called **offsets.**

If AC contains d units of length, and BX, DY p and q units respectively,

the area of the quad[l] ABCD $= \triangle ABC + \triangle ADC$

$$= \frac{1}{2} AC \cdot BX + \frac{1}{2} AC \cdot DY$$

$$= \frac{1}{2} dp + \frac{1}{2} dq = \frac{1}{2} d(p+q).$$

That is to say,

the area of a quadrilateral $= \frac{1}{2}$ *diagonal* $\times$ (*sum of offsets*).

EXERCISES.

(Numerical and Graphical.)

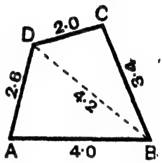
1. Find the area of the trapezium in which the two parallel sides are 4·7″ and 3·3″, and the height 1·5″.

2. In a quadrilateral ABCD, the diagonal AC = 17 feet; and the offsets from it to B and D are 11 feet and 9 feet. Find the area.

3. In a plan ABCD of a quadrilateral enclosure, the diagonal AC measures 8·2 cm., and the offsets from it to B and D are 3·4 cm. and 2·6 cm. respectively. If 1 cm. in the plan represents 5 metres, find the area of the enclosure.

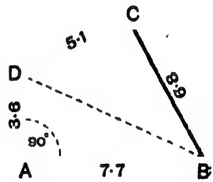
4. Draw a quadrilateral ABCD from the adjoining rough plan, the dimensions being given in inches.

Draw and measure the offsets to A and C from the diagonal BD; and hence calculate the area of the quadrilateral.



5. Draw a quadrilateral ABCD from the details given in the adjoining plan. The dimensions are to be in centimetres.

Make any necessary measurements of your figure, and calculate its area.



6. Draw a trapezium ABCD from the following data: AB and CD are the parallel sides. AB = 4″; AD = BC = 2″; the ∠A = the ∠B = 60°.

Make any necessary measurements, and calculate the area.

7. Draw a trapezium ABCD in which AB and CD are the parallel sides; and AB = 9 cm., CD = 3 cm., and AD = BC = 5 cm.

Make any necessary measurement, and calculate the area.

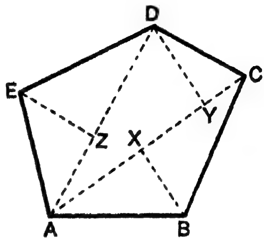
8. From the formula *area of quad*^{l} = $\frac{1}{2}$ *diag.* × (*sum of offsets*) shew that, if the diagonals are *at right angles*,

$$area = \tfrac{1}{2}\,(product\ of\ diagonals).$$

9. Given the lengths of the diagonals of a quadrilateral, and the angle between them, prove that the area is the same wherever they intersect.

THE AREA OF ANY RECTILINEAL FIGURE.

1[st] METHOD. A rectilineal figure may be divided into triangles whose areas can be separately calculated from suitable measurements. The sum of these areas will be the area of the given figure.

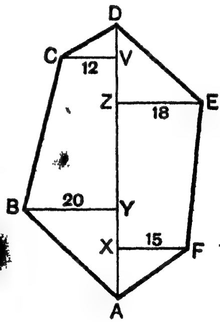


Example. The measurements required to find the area of the figure ABCDE are AC, AD, and the offsets BX, DY, EZ.

2[nd] METHOD. The area of a rectilineal figure is also found by taking a **base-line** (AD in the diagram below) and offsets from it. These divide the figure into *right-angled* triangles and *right-angled* trapeziums, whose areas may be found after measuring the offsets and the various sections of the base-line.

Example. Find the area of the enclosure ABCDEF from the plan and measurements tabulated below.

	YARDS.	
	AD = 56	
VC = 12	AV = 50	
	AZ = 40	ZE = 18
YB = 20	AY = 18	
	AX = 10	XF = 15



The measurements are made from A along the base line to the points from which the offsets spring.

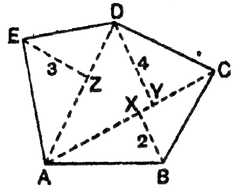
		Sq. yds.	Sq. yds.
Here	$\triangle$ AXF = $\frac{1}{2}$. AX × XF	= $\frac{1}{2}$ × 10 × 15 =	75
	$\triangle$ AYB = $\frac{1}{2}$. AY × YB	= $\frac{1}{2}$ × 18 × 20 =	180
	$\triangle$ DZE = $\frac{1}{2}$. DZ × ZE	= $\frac{1}{2}$ × 16 × 18 =	144
	$\triangle$ DVC = $\frac{1}{2}$. DV × VC	= $\frac{1}{2}$ × 6 × 12 =	36
	trap[m] XFEZ = $\frac{1}{2}$. XZ × (XF + ZE)	= $\frac{1}{2}$ × 30 × 33 =	495
	trap[m] YBCV = $\frac{1}{2}$. YV × (YB + VC)	= $\frac{1}{2}$ × 32 × 32 =	512

$\therefore$, by addition, the fig. ABCDEF = 1442 sq. yds.

EXERCISES.

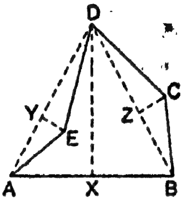
1. Calculate the areas of the figures (i) and (ii) from the plans and dimensions (in cms.) given below.

(i)



AC = 6 cm., AD = 5 cm.
Lengths of offsets figured in diagram.

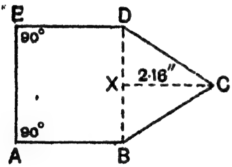
(ii)



AB = BD = DA = 6 cm.
EY = CZ = 1 cm.
DX = 5·2 cm.

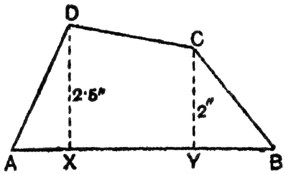
2. Draw full size the figures whose plans and dimensions are given below; and calculate the area in each case.

(i)



The fig. is equilateral: each side to be $2\frac{1}{2}''$.

(ii)

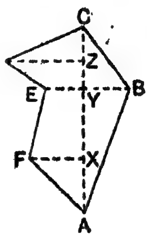


AX = $1\frac{1}{4}''$, XY = $2\frac{3}{4}''$,
YB = $1\frac{3}{4}''$.

3. Find the area of the figure ABCDEF from the following measurements and draw a plan in which 1 cm. represents 20 metres.

	Metres.	
	to C	
	180	
80 to D	150	
40 to E	120	50 to B
60 to F	50	
	From A	

The Plan.



EXERCISES ON QUADRILATERALS.

(Theoretical.)

1. ABCD is a rectangle, and PQRS the figure formed by joining in order the middle points of the sides.

Prove (i) that PQRS is a rhombus;

(ii) that the area of PQRS is half that of ABCD.

Hence shew that *the area of a rhombus is half the product of its diagonals.*

Is this true of any quadrilateral whose diagonals cut at right angles? Illustrate your answer by a diagram.

2. Prove that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals.

Hence shew how a parallelogram ABCD may be bisected by a straight line drawn

(i) through a given point P;

(ii) perpendicular to the side AB;

(iii) parallel to a given line QR.

3. In the trapezium ABCD, AB is parallel to DC; and X is the middle point of BC. Through X draw PQ parallel to AD to meet AB and DC produced at P and Q. Then prove

(i) trapezium ABCD $=$ par^{m} APQD.

(ii) trapezium ABCD $=$ twice the $\triangle$AXD.

(Graphical.)

4. The diagonals of a quadrilateral ABCD cut at right angles, and measure $3{\cdot}0''$ and $2{\cdot}2''$ respectively. Find the area.

Shew by a figure that the area is the same wherever the diagonals cut, so long as they are at right angles.

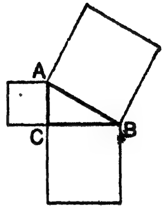
5. In the parallelogram ABCD, AB $=8{\cdot}0$ cm., AD $=3{\cdot}2$ cm., and the perpendicular distance between AB and DC $=3{\cdot}0$ cm. Draw the parallelogram. Calculate the distance between AD and BC; and check your result by measurement.

6. One side of a parallelogram is $2{\cdot}5''$, and its diagonals are $3{\cdot}4''$ and $2{\cdot}4''$. Construct the parallelogram; and, after making any necessary measurement, calculate the area.

7. ABCD is a parallelogram on a fixed base AB and of constant area. Find the locus of the intersection of its diagonals.

EXERCISES LEADING TO THEOREM 29.

In the adjoining diagram, ABC is a triangle *right-angled* at C; and squares are drawn on the three sides. Let us compare the area of the square on the hypotenuse AB with the *sum* of the squares on the sides AC, CB which contain the right angle.



1. Draw the above diagram, making AC=3 cm., and BC=4 cm.;

$$\left.\begin{array}{l}\text{Then the area of the square on AC} = 3^2, \text{ or } \ 9 \text{ sq. cm.}\\ \text{and} \ \ldots\ldots\ldots\ldots \text{ the square on BC} = 4^2, \text{ or } 16 \text{ sq. cm.}\end{array}\right\}$$

$\therefore$ the sum of the squares on AC, BC = 25 sq. cm.

Now *measure* AB; hence calculate the area of the square on AB, and compare the result with the *sum* already obtained.

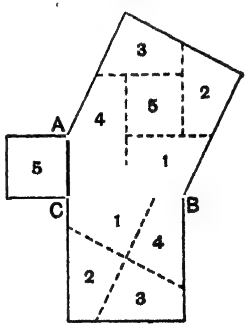
2. Repeat the process of the last exercise, making AC=1·0″, and BC=2·4″.

3. If $a=15$, $b=8$, $c=17$, shew arithmetically that $c^2=a^2+b^2$.

Now draw on squared paper a triangle ABC, whose sides a, b, and c are 15, 8, and 17 units of length; and *measure the angle* ACB.

4. Take any triangle ABC, right-angled at C; and draw squares on AC, CB, and on the hypotenuse AB.

Through the mid-point of the square on CB (*i.e.* the intersection of the diagonals) draw lines parallel and perpendicular to the hypotenuse, thus dividing the square into four *congruent* quadrilaterals. These, together with the square on AC, will be found exactly to fit into the square on AB, in the way indicated by corresponding numbers.



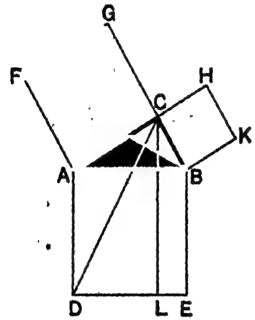
These experiments point to the conclusion that:

In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

A formal proof of this theorem is given on the next page.

THEOREM 29. [Euclid I. 47.]

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.



Let ABC be a right-angled $\triangle$, having the angle ACB a rt. $\angle$.

It is required to prove that the square on the hypotenuse AB = *the sum of the squares on* AC, CB.

On AB describe the sq. ADEB; and on AC, CB describe the sqq. ACGF, CBKH.

Through C draw CL par[l] to AD or BE.

Join CD, FB.

Proof. Because each of the $\angle^s$ ACB, ACG is a rt. $\angle$,

$\therefore$ BC and CG are in the same st. line.

Now the rt. $\angle$ BAD = the rt. $\angle$ FAC;

add to each the $\angle$ CAB:

then the whole $\angle$ CAD = the whole $\angle$ FAB.

Then in the $\triangle^s$ CAD, FAB,

because $\begin{cases} \text{CA} = \text{FA}, \\ \text{AD} = \text{AB}, \\ \text{and the included } \angle \text{ CAD} = \text{the included } \angle \text{ FAB}; \end{cases}$

$\therefore$ the $\triangle$ CAD = the $\triangle$ FAB. *Theor.* 4.

Now the rect. AL is double of the $\triangle$CAD, being on the same base AD, and between the same par[ls] AD, CL.

And the sq. GA is double of the $\triangle$FAB, being on the same base FA, and between the same par[ls] FA, GB.

$\therefore$ the rect. AL = the sq. GA.

Similarly by joining CE, AK, it can be shewn that
the rect. BL = the sq. HB.

$\therefore$ the whole sq. AE = the sum of the sqq. GA, HB:

that is, the square on the hypotenuse AB = the sum of the squares on the two sides AC, CB.

Q.E.D.

Obs. This is known as the Theorem of Pythagoras. The result established may be stated as follows:

$$AB^2 = BC^2 + CA^2.$$

That is, if a and b denote the lengths of the sides containing the right angle; and if c denotes the hypotenuse,

$$c^2 = a^2 + b^2.$$

Hence $\quad a^2 = c^2 - b^2$; and $b^2 = c^2 - a^2$.

NOTE 1. The following important results should be noticed.

If CL and AB intersect in O, it has been shewn in the course of the proof that

the sq. GA = the rect. AL;
that is, AC^2 = the rect. contained by AB, AO. ....(i)

Also the sq. HB = the rect. BL;
that is, BC^2 = the rect. contained by BA, BO. ....(ii)

NOTE 2. It can be proved by superposition that *squares standing on equal sides are equal in area.*

Hence we conclude, conversely,

If two squares are equal in area they stand on equal sides.

EXPERIMENTAL PROOFS OF PYTHAGORAS'S THEOREM.

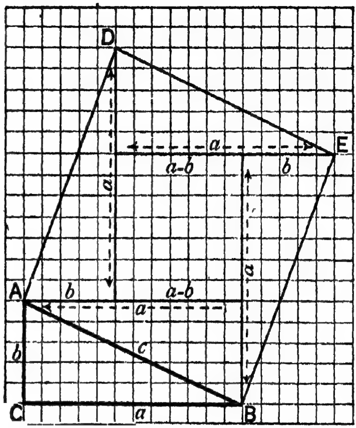
I. Here ABC is the given rt.-angled Δ; and ABED is the square on the hypotenuse AB.

By drawing lines par[l] to the sides BC, CA, it is easily seen that the sq. BD is divided into 4 rt.-angled Δ^s, each identically equal to ABC, together with a central square.

Hence

sq. on hypotenuse $c = 4$ rt. $\angle^d$ Δ^s + the central square

$$= 4 \cdot \tfrac{1}{2}ab + (a-b)^2$$
$$= 2ab + a^2 - 2ab + b^2$$
$$= a^2 + b^2.$$

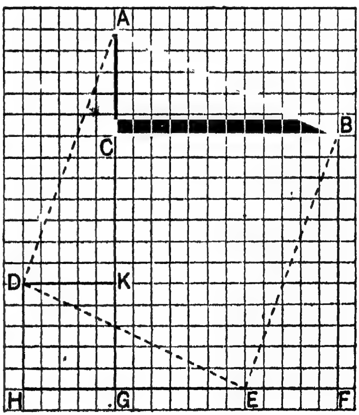


II. Here ABC is the given rt.-angled Δ, and the figs. CF, HK are the sqq. on CB, CA placed side by side.

FE is made equal to DH or CA; and the two sqq. CF, HK are cut along the lines BE, ED.

Then it will be found that the Δ DHE may be placed so as to fill up the space ACB; and the Δ BFE may be made to fill the space AKD.

Hence the two sqq. CF, HK may be fitted together so as to form the single fig. ABED, which will be found to be a perfect square, namely the square on the hypotenuse AB.



EXERCISES.

(*Numerical and Graphical.*)

Draw a triangle ABC, right-angled at C, having given:

(i) $a=3$ cm., $b=4$ cm.;

(ii) $a=2\cdot5$ cm., $b=6\cdot0$ cm.;

(iii) $a=1\cdot2''$, $b=3\cdot5''$.

In each case calculate the length of the hypotenuse c, and verify your result by measurement.

2. Draw a triangle ABC, right-angled at C having given:

(i) $c=3\cdot4''$, $a=3\cdot0''$; [See Problem 10]

(ii) $c=5\cdot3$ cm., $b=4\cdot5$ cm.

In each case calculate the remaining side, and verify your result by measurement.

(*The following examples are to be solved by calculation; but in each case a plan should be drawn on some suitable scale, and the calculated result verified by measurement.*)

3. A ladder whose foot is 9 feet from the front of a house reaches to a window-sill 40 feet above the ground. What is the length of the ladder?

4. A ship sails 33 miles due South, and then 56 miles due West How far is it then from its starting point?

5. Two ships are observed from a signal station to bear respectively N.E. 6·0 km. distant, and N.W. 1·1 km. distant. How far are they apart?

6. A ladder 65 feet long reaches to a point in the face of a house 63 feet above the ground. How far is the foot from the house?

7. B is due East of A, but at an unknown distance. C is due South of B, and distant 55 metres. AC is known to be 73 metres. Find AB.

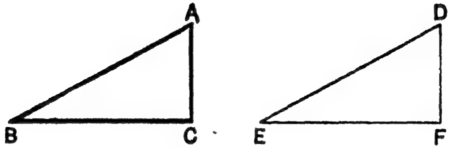
8. A man travels 27 miles due South; then 24 miles due West; finally 20 miles due North. How far is he from his starting point?

9. From A go West 25 metres, then North 60 metres, then East 80 metres, finally South 12 metres. How far are you then from A?

10. A ladder 50 feet long is placed so as to reach a window 48 feet high; and on turning the ladder over to the other side of the street, it reaches a point 14 feet high. Find the breadth of the street.

THEOREM 30. [Euclid I. 48.]

If the square described on one side of a triangle is equal to the sum of the squares described on the other two sides, then the angle contained by these two sides is a right angle.



Let ABC be a triangle in which
the sq. on AB = the sum of the sqq. on BC, CA.

It is required to prove that ACB *is a right angle.*

Make EF equal to BC.
Draw FD perp^r to EF, and make FD equal to CA.
Join ED.

Proof. Because EF = BC,
∴ the sq. on EF = the sq. on BC.
And because FD = CA,
∴ the sq. on FD = the sq. on CA.

Hence the sum of the sqq. on EF, FD = the sum of the sqq. on BC, CA.

But since EFD is a rt. ∠,
∴ the sum of the sqq. on EF, FD = the sq. on DE: *Theor.* 29.
And, by hypothesis, the sqq. on BC, CA = the sq. on AB.
∴ the sq. on DE = the sq. on AB.
∴ DE = AB.

Then in the △^{s} ACB, DFE,

$$\text{because}\left\{\begin{array}{l} \text{AC} = \text{DF}, \\ \text{CB} = \text{FE}, \\ \text{and AB} = \text{DE}; \end{array}\right.$$

∴ the ∠ACB = the ∠DFE. *Theor.* 7

But, by construction, DFE is a right angle;
∴ the ∠ACB is a right angle.

Q.E.D.

EXERCISES ON THEOREMS 29, 30.

(Theoretical.)

1. Shew that the square on the diagonal of a given square is double of the given square.

2. In the $\triangle ABC$, AD is drawn perpendicular to the base BC. If the side c is greater than b,

shew that $c^2 - b^2 = BD^2 - DC^2$.

3. If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to BC, CA, AB respectively: shew that

$$AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2.$$

4. ABC is a triangle right-angled at A; and the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined. Prove that

$$BQ^2 + PC^2 = BC^2 + PQ^2.$$

5. In a right-angled triangle four times the sum of the squares on the medians drawn from the acute angles is equal to five times the square on the hypotenuse.

6. Describe a square equal to the sum of two given squares.

7. Describe a square equal to the difference between two given squares.

8. Divide a straight line into two parts so that the square on one part may be twice the square on the other.

9. Divide a straight line into two parts such that the sum of their squares shall be equal to a given square.

(Numerical and Graphical.)

10. Determine which of the following triangles are right-angled:

(i) $a = 14$ cm., $b = 48$ cm., $c = 50$ cm.;
(ii) $a = 40$ cm., $b = 10$ cm., $c = 41$ cm.;
(iii) $a = 20$ cm., $b = 99$ cm., $c = 101$ cm.

11. ABC is an isosceles triangle right-angled at C; deduce from Theorem 29 that

$$AB^2 = 2AC^2.$$

Illustrate this result graphically by drawing both diagonals of the square on AB, and one diagonal of the square on AC.

If $AC = BC = 2''$, find AB to the nearest hundredth of an inch, and verify your calculation by actual construction and measurement.

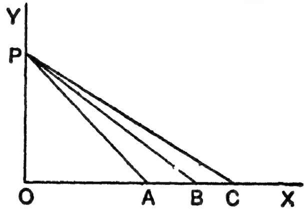
12. Draw a square on a diagonal of 6 cm. Calculate, and also measure, the length of a side. Find the area.

PROBLEM 16.

To draw squares whose areas shall be respectively twice, three-times, four-times, ..., that of a given square.

Hence find graphically approximate values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, ...

Take OX, OY at right angles to one another, and from them mark off OA, OP, each *one* unit of length. Join PA.



$$\text{Then } PA^2 = OP^2 + OA^2 = 1 + 1 = 2.$$

$$\therefore\ PA = \sqrt{2}.$$

From OX mark off OB equal to PA, and join PB;

$$\text{then } PB^2 = OP^2 + OB^2 = 1 + 2 = 3.$$

$$\therefore\ PB = \sqrt{3}.$$

From OX mark off OC equal to PB, and join PC;

$$\text{then } PC^2 = OP^2 + OC^2 = 1 + 3 = 4.$$

$$\therefore\ PC = \sqrt{4}.$$

The lengths of PA, PB, PC may now be found *by measurement*; and by continuing the process we may find $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, ...

EXERCISES ON THEOREMS 29, 30 (*Continued*).

13. Prove the following formula:

$$\textit{Diagonal of square} = \textit{side} \times \sqrt{2}.$$

Hence find to the nearest centimetre the diagonal of a square on a side of 50 metres.

Draw a plan (scale 1 cm. to 10 metres) and obtain the result as nearly as you can by measurement.

14. ABC is an equilateral triangle of which each side $=2m$ units, and the perpendicular from any vertex to the opposite side $=p$.

Prove that $p = m\sqrt{3}$.

Test this result graphically, when each side $=8$ cm.

15. If in a triangle $a=m^2-n^2$, $b=2mn$, $c=m^2+n^2$; prove algebraically that $c^2=a^2+b^2$.

Hence by giving various numerical values to m and n, find sets of numbers representing the sides of right-angled triangles.

16. In a triangle ABC, AD is drawn perpendicular to BC. Let p denote the length of AD.

(i) If $a=25$ cm., $p=12$ cm., BD$=9$ cm.; find b and c.

(ii) If $b=41''$, $c=50''$, BD$=30''$; find p and a.

And prove that $\sqrt{b^2-p^2}+\sqrt{c^2-p^2}=a$.

17. In the triangle ABC, AD is drawn perpendicular to BC. Prove that

$$c^2-\text{BD}^2=b^2-\text{CD}^2.$$

If $a=51$ cm., $b=20$ cm., $c=37$ cm.; find BD.

Thence find p, the length of AD, and the area of the triangle ABC.

18. Find by the method of the last example the areas of the triangles whose sides are as follows:

(i) $a=17''$, $b=10''$, $c=9''$. (ii) $a=25$ ft., $b=17$ ft., $c=12$ ft.
(iii) $a=41$ cm., $b=28$ cm., $c=15$ cm. (iv) $a=40$ yd., $b=37$ yd., $c=13$ yd.

19. A straight rod PQ slides between two straight rulers OX, OY placed at right angles to one another. In one position of the rod OP$=5{\cdot}6$ cm., and OQ$=3{\cdot}3$ cm. If in another position OP$=4{\cdot}0$ cm., find OQ graphically; and test the accuracy of your drawing by calculation.

20. ABC is a triangle right-angled at C, and p is the length of the perpendicular from C on AB. By expressing the area of the triangle in two ways, shew that

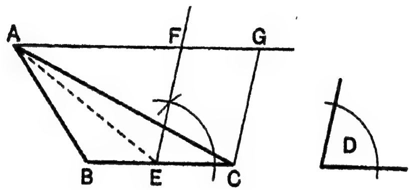
$$pc=ab.$$

Hence deduce
$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

PROBLEMS ON AREAS.

PROBLEM 17.

To describe a parallelogram equal to a given triangle, and having one of its angles equal to a given angle.



Let ABC be the given triangle, and D the given angle.

It is required to describe a parallelogram equal to ABC, *and having one of its angles equal to* D.

Construction. Bisect BC at E.
At E in CE, make the ∠ CEF equal to D;
through A draw AFG par[l] to BC;
and through C draw CG par[l] to EF.
Then FECG is the required par[m].

Proof. Join AE.
Now the △[s] ABE, AEC are on equal bases BE, EC, and of the same altitude;

∴ the △ ABE = the △ AEC.
∴ the △ ABC is double of the △ AEC.

But FECG is a par[m] by construction;
and it is double of the △ AEC,
being on the same base EC, and between the same par[ls] EC and AG.

∴ the par[m] FECG = the △ ABC;
and one of its angles, namely CEF, = the given ∠ D.

EXERCISES.

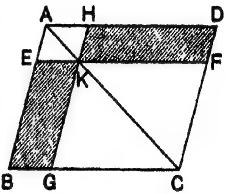
(Graphical.)

1. Draw a square on a side of 5 cm., and make a parallelogram of equal area on the same base, and having an angle of 45°.

Find (i) by calculation, (ii) by measurement the length of an oblique side of the parallelogram.

2. Draw any parallelogram ABCD in which AB = $2\frac{1}{2}''$ and AD = $2''$; and on the base AB draw a rhombus of equal area.

DEFINITION. In a parallelogram ABCD, if through any point K in the diagonal AC parallels EF, HG are drawn to the sides, then the figures EH, GF are called **parallelograms about** AC, and the figures EG, HF are said to be their **complements.**



3. *In the diagram of the preceding definition shew by Theorem* 21 *that the complements* EG, HF *are equal in area.*

Hence, given a parallelogram EG, and a straight line HK, deduce a construction for drawing on HK as one side a parallelogram equal and equiangular to the parallelogram EG.

4. Construct a rectangle equal in area to a given rectangle CDEF, and having one side equal to a given line AB.

If AB = 6 cm., CD = 8 cm., CF = 3 cm., find by measurement the remaining side of the constructed rectangle.

5. Given a parallelogram ABCD, in which AB = $2{\cdot}4''$, AD = $1{\cdot}8''$, and the $\angle$A = 55°. Construct an equiangular parallelogram of equal area, the greater side measuring $2{\cdot}7''$. Measure the shorter side.

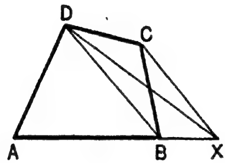
Repeat the process giving to A any other value; and compare your results. What conclusion do you draw?

6. Draw a rectangle on a side of 5 cm. equal in area to an equilateral triangle on a side of 6 cm.

Measure the remaining side of the rectangle, and calculate its approximate area.

PROBLEM 18.

To draw a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral.

It is required to describe a triangle equal to ABCD *in area.*

Construction. Join DB.
Through C draw CX par[l] to DB, meeting AB produced in X.
Join DX.
Then DAX is the required triangle.

Proof. Now the $\triangle^s$ XDB, CDB are on the same base DB and between the same par[ls] DB, CX;

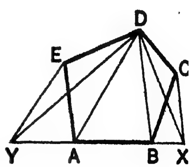
$\therefore$ the $\triangle$ XDB = the $\triangle$ CDB in area.

To each of these equals add the $\triangle$ ADB;
then the $\triangle$ DAX = the fig. ABCD.

COROLLARY. In the same way it is always possible to draw a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure; and thus step by step, any rectilineal figure may be reduced to a triangle of equal area.

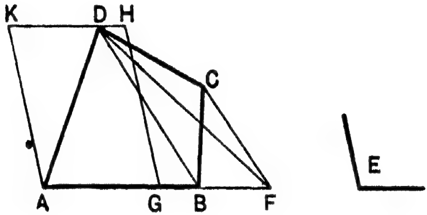
For example, in the adjoining diagram the five-sided fig. EDCBA is equal in area to the four-sided fig. EDXA.

The fig. EDXA may now be reduced to an equal $\triangle$ DXY.



PROBLEM 19.

To draw a parallelogram equal in area to a given rectilineal figure, and having an angle equal to a given angle.



Let ABCD be the given rectil. fig., and E the given angle.

It is required to draw a par^m equal to ABCD *and having an angle equal to* E.

Construction. Join DB.

Through C draw CF par^l to DB, and meeting AB produced in F.

Join DF.

Then the $\triangle$ DAF = the fig. ABCD. *Prob.* 18.

Draw the par^m AGHK equal to the $\triangle$ ADF, and having the $\angle$ KAG equal to the $\angle$ E. *Prob.* 17.

Then the par^m KG = the $\triangle$ ADF
= the fig. ABCD;
and it has the $\angle$ KAG equal to the $\angle$ E.

NOTE. If the given rectilineal figure has more than four sides, it must first be reduced, step by step, until it is replaced by an equivalent triangle.

EXERCISES.

(Reduction of a Rectilineal Figure to an equivalent Triangle.)

Draw a quadrilateral ABCD from the following data:

AB = BC = 5·5 cm.; CD = DA = 4·5 cm.; the $\angle$A = 75°.

Reduce the quadrilateral to a triangle of equal area. Measure the base and altitude of the triangle; and hence calculate the approximate area of the given figure.

2. Draw a quadrilateral ABCD having given:
AB = 2·8″, BC = 3·2″, CD = 3·3″, DA = 3·6″, and the diagonal BD = 3·0″.
Construct an equivalent triangle; and hence find the approximate area of the quadrilateral.

3. On a base AB, 4 cm. in length, describe an equilateral pentagon (5 sides), having each of the angles at A and B 108°.
Reduce the figure to a triangle of equal area; and by measuring its base and altitude, calculate the approximate area of the pentagon.

4. A quadrilateral field ABCD has the following measurements:
AB = 450 metres, BC = 380 m., CD = 330 m., AD = 390 m.,
and the diagonal AC = 660 m.
Draw a plan (scale 1 cm. to 50 metres). Reduce your plan to an equivalent triangle, and measure its base and altitude. Hence estimate the area of the field.

(Problems. State your construction, and give a theoretical proof.)

5. Reduce a triangle ABC to a triangle of equal area having its base BD of given length. (D lies in BC, or BC produced.)

6. Construct a triangle equal in area to a given triangle, and having a given altitude.

7. ABC is a given triangle, and X a given point. Draw a triangle equal in area to ABC, having its vertex at X, and its base in the same straight line as BC.

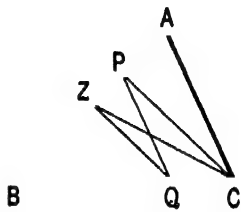
8. Construct a triangle equal in area to the quadrilateral ABCD, having its vertex at a given point X in DC, and its base in the same straight line as AB.

9. Shew how a triangle may be divided into n equal parts by straight lines drawn through one of its angular points.

10. *Bisect a triangle by a straight line drawn through a given point in one of its sides.*

[Let ABC be the given △, and P the given point in the side AB.

Bisect AB at Z; and join CZ, CP.
Through Z draw ZQ parallel to CP.
Join PQ.
Then PQ bisects the △.]



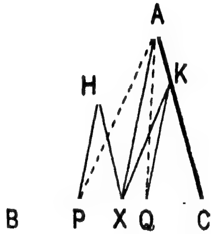
11. *Trisect a triangle by straight lines drawn from a given point in one of its sides.*

[Let ABC be the given △, and X the given point in the side BC.

Trisect BC at the points P, Q. *Prob.* 7.
Join AX, and through P and Q draw PH and QK parallel to AX.

Join XH, XK.

These straight lines trisect the △; as may be shewn by joining AP, AQ.]



12. Cut off from a given triangle a fourth, fifth, sixth, or any part required by a straight line drawn from a given point in one of its sides.

13. *Bisect a quadrilateral by a straight line drawn through an angular point.*

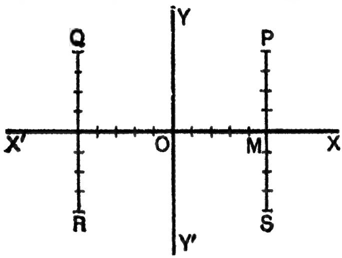
[Reduce the quadrilateral to a triangle of equal area, and join the vertex to the middle point of the base.]

14. Cut off from a given quadrilateral a third, a fourth, a fifth, or any part required, by a straight line drawn through a given angular point.

AXES OF REFERENCE. COORDINATES.

EXERCISES FOR SQUARED PAPER.

If we take two fixed straight lines XOX′, YOY′ cutting one another at right angles at O, the position of any point P with reference to these lines is known when we know its distances from each of them. Such lines are called **axes of reference**, XOX′ being known as the **axis of** x, and YOY′ as the **axis of** y. Their point of intersection O is called the **origin**.



The lines XOX′, YOY′ are usually drawn horizontally and vertically.

In practice the distances of P from the axes are estimated thus:

From P, PM is drawn perpendicular to X′X; and OM and PM are measured.

OM is called the **abscissa** of the point P, and is denoted by x.
PM „ **ordinate** „ „ „ y.

The abscissa and ordinate taken together are called the **coordinates** of the point P, and are denoted by (x, y).

We may thus find the position of a point if its coordinates are known.

EXAMPLE. *Plot the point whose coordinates are* (5, 4).
Along OX mark off OM, 5 units in length.
At M draw MP perp. to OX, making MP 4 units in length.
Then P is the point whose coordinates are (5, 4).

The **axes** of reference divide the plane into four regions XOY, YOX′, X′OY′, Y′OX, known respectively as the first, second, third, and fourth **quadrants**.

It is clear that in each quadrant there is a point whose distances from the axes are equal to those of P in the above diagram, namely, 5 units and 4 units.

The coordinates of these points are distinguished by the use of the *positive* and *negative* signs, according to the following system:

Abscissæ measured along the x-axis to the right of the origin are positive, those measured to the left of the origin are negative. Ordinates which lie above the x-axis (that is, in the first and second quadrants) are positive; those which lie below the x-axis (that is, in the third and fourth quadrants) are negative

Thus the coordinates of the points Q, R, S are

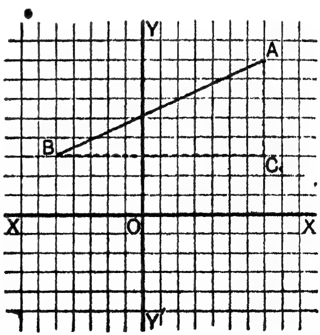
$$(-5, 4),\ (-5, -4),\ \text{and}\ (5, -4)\ \text{respectively.}$$

NOTE. The coordinates of the origin are $(0, 0)$.

In practice it is convenient to use squared paper. Two intersecting lines should be chosen as axes, and slightly thickened to aid the eye, then one or more of the length divisions may be taken as the linear unit. The paper used in the following examples is ruled to tenths of an inch.

EXAMPLE 1. *The coordinates of the points* A *and* B *are* $(7, 8)$ *and* $(-5, 3)$: *plot the points and find the distance between them.*

After plotting the points as in the diagram, we may find AB approximately by direct measurement.



Or we may proceed thus:

Draw through B a line par[l] to XX′ to meet the ordinate of A at C. Then ACB is a rt.-angled △ in which BC = 12, and AC = 5.

Now $AB^2 = BC^2 + AC^2$

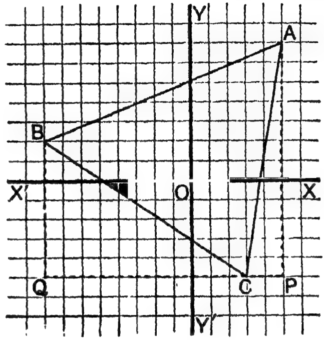
$= 12^2 + 5^2$

$= 144 + 25$

$= 169.$

$\therefore\ AB = 13.$

EXAMPLE 2. *The coordinates of* A, B, *and* C *are* (5, 7), (−8, 2), *and* (3, −5); *plot these points and find the area of the triangle of which these are the vertices.*



Having plotted the points as in the diagram, we may measure AB, and draw and measure the perp. from C on AB. Hence the approximate area may be calculated.

Or we may proceed thus:

Through A and B draw AP, BQ par[l] to YY'.

Through C draw PQ par[l] to XX'.

Then the △ ABC = the trap[m] APQB − the two rt.-angled △[s] APC, BQC

$= \frac{1}{2}PQ(AP+BQ) - \frac{1}{2}.AP.PC - \frac{1}{2}.BQ.QC$

$= \frac{1}{2}\times 13\times 19 \quad - \frac{1}{2}\times 12\times 2 \quad - \frac{1}{2}\times 7\times 11$

$= 73$ units of area.

EXERCISES.

1. Plot the following sets of points:

(i) (6, 4), (−6, 4), (−6, −4), (6, −4);

(ii) (8, 0), (0, 8), (−8, 0), (0, −8);

(iii) (12, 5), (5, 12), (−12, 5), (−5, 12).

2. Plot the following points, and shew experimentally that each set lie in one straight line.

(i) (9, 7), (0, 0), (−9, −7); (ii) (−9, 7), (0, 0), (9, −7).

Explain these results theoretically.

3. Plot the following pairs of points; join the points in each case, and measure the coordinates of the mid-point of the joining line.

(i) (4, 3), (12, 7); (ii) (5, 4), (15, 16).

Shew *why* in each case the coordinates of the mid-point are respectively *half the sum of the abscissæ* and *half the sum of the ordinates* of the given points.

4. Plot the following pairs of points; and find the coordinates of the mid-point of their joining lines.

(i) (0, 0), (8, 10); (ii) (8, 0), (0, 10);

(iii) (0, 0), (−8, −10); (iv) (−8, 0), (0, −10).

5. Find the coordinates of the points of trisection of the line joining (0, 0) to (18, 15).

6. Plot the two following series of points:

(i) (5, 0), (5, 2), (5, 5), (5, -1), (5, -4);

(ii) (-4, 8), (-1, 8), (0, 8), (3, 8), (6, 8).

Shew that they lie on two lines respectively parallel to the axis of y, and the axis of x. Find the coordinates of the point in which they intersect.

7. Plot the following points, and calculate their distances from the origin.

(i) (15, 8); (ii) (-15, -8); (iii) (2·4″, ·7″); (iv) ($-$·7″, 2·4″).

Check your results by measurement.

8. Plot the following pairs of points, and in each case calculate the distance between them.

(i) (4, 0), (0, 3); (ii) (9, 8), (5, 5);

(iii) (15, 0), (0, 8); (iv) (10, 4), (-5, 12);

(v) (20, 12), (-15, 0); (vi) (20, 9), (-15, -3).

Verify your calculation by measurement.

9. Shew that the points (-3, 2), (3, 10), (7, 2) are the angular points of an isosceles triangle. Calculate and measure the lengths of the equal sides.

10. Plot the eight points (0, 5), (3, 4), (5, 0), (4, -3), (-5, 0), (0, -5), (-4, 3), (-4, -3), and shew that they all lie on a circle whose centre is the origin.

11. Explain by a diagram why the distances between the following pairs of points are all equal.

(i) (a, 0), (0, b); (ii) (b, 0), (0, a); (iii) (0, 0), (a, b).

12. Draw the straight lines joining

(i) (a, 0) and (0, a); (ii) (0, 0) and (a, a);

and prove that these lines bisect each other at right angles.

13. Shew that (0, 4), (12, 9), (12, -4) are the vertices of an isosceles triangle whose base is bisected by the axis of x.

14. Three vertices of a rectangle are (14, 0), (14, 10), and (0, 10) find the coordinates of the fourth vertex, and of the intersection of the diagonals.

15. Prove that the four points (0, 0), (13, 0), (18, 12), (5, 12) are the angular points of a rhombus. Find the length of each side, and the coordinates of the intersection of the diagonals.

16. Plot the locus of a point which moves so that its distances from the points (0, 0) and (4, -4) are always equal to one another. Where does the locus cut the axes?

17. Shew that the following groups of points are the vertices of rectangles. Draw the figures, and calculate their areas.

(i) (4, 3), (17, 3), (17, 12), (4, 12);
(ii) (3, 2), (3, 15), (−6, 15), (−6, 2);
(iii) (5, 1), (−8, 1), (−8, −8), (5, −8).

18. Join in order the points (1″, 0), (0, 1″), (−1″, 0), (0, −1″). Of what kind is the quadrilateral so formed? Find its area.

If a second figure is formed by joining the middle points of the first, find its area.

19. Plot the triangles given by the following sets of points; and find their areas.

(i) (10, 10), (4, 0), (18, 0); (ii) (10, −10), (4, 0), (18, 0);
(iii) (−10, 10), (−4, 0), (−18, 0); (iv) (−10, −10), (−4, 0), (−18, 0).

20. Draw the triangles given by the points

(i) (0, 0), (5, 3), (6, 0); (ii) (0, 0), (3, 0), (0, 6).

Find their areas; and measure the angles of the first triangle.

21. Plot the triangles given by the following sets of points. Shew that in each case one side is parallel to one of the axes. Hence find the area.

(i) (0, 0), (12, 10), (12, −6); (ii) (0, 0), (5, 8), (−15, 8);
(iii) (0, 0), (−12, 12), (−12, −8); (iv) (0, 0), (−6, −8), (20, −8).

22. In the following triangles shew that two sides of each are parallel to the axes. Find their areas.

(i) (5, 5), (15, 5), (15, 15); (ii) (8, 3), (8, 18), (0, 18);
(iii) (4, 8), (−16, −4), (4, −4); (iv) (1, 15), (−11, 15), (1, −7).

23. Shew that (−5, 5), (7, 10), (10, 6), (−2, 1) are the angular points of a parallelogram. Find its sides and area.

24. Shew that each of the following sets of points gives a trapezium. Find the area of each.

(i) (3, 0), (3, 3), (9, 0), (9, 6); (ii) (0, 3), (−5, 3), (−2, −3), (0, −3);
(iii) (8, 4), (4, 4), (11, −1), (3, −1); (iv) (0, 0), (−1, 5), (−4, 5), (−8, 0).

25. Find the area of the triangles given by the following points:

(i) (5, 5), (20, 10), (12, 14); (ii) (7, 6), (−10, 4), (−4, −3);
(iii) (0, −6), (0, −3), (14, 5); (iv) (6, 4), (−7, −6), (−2, −15).

26. Shew that (−5, 0), (7, 5), (19, 0), (7, −5) are the angular points of a rhombus. Find its sides and its area.

27. Join the points $(0, -5)$, $(12, 0)$, $(4, 6)$, $(-8, -3)$, in the order given. Calculate the lengths of the first three sides and measure the fourth. Find the areas of the portions of the figure lying in the first and fourth quadrants.

28. The coordinates of four points A, B, C, D are respectively

$$(-4, -4),\ (-10, 4),\ (-10, 13),\ (5, 5).$$

Calculate the lengths of AB, BC, CD, and measure AD. Also calculate the area of ABCD by considering it as the difference of two triangles.

29. Draw the figure whose angular points are given by

$$(0, -3),\ (8, 3),\ (-4, 8),\ (-4, 3),\ (0, 0).$$

Find the lengths of its sides, taking the points in the above order. Also divide it into three right-angled triangles, and hence find its area.

30. A plan of a triangular field ABC is drawn on squared paper (scale $1'' = 100$ yds.). On the plan the coordinates of A, B, C are $(-1'', -3'')$, $(3'', 4'')$, $(-5'', -2'')$ respectively. Find the area of the field, the length of the side represented by BC, and the distance from this side of the opposite corner of the field.

31. Shew that the points $(6, 0)$, $(20, 6)$, $(14, 20)$, $(0, 14)$ are the vertices of a square. Measure a side and hence find the approximate area. Calculate the area exactly (i) by drawing a circumscribing square through its vertices; (ii) by subdividing the given square as in the first figure on page 120.

MISCELLANEOUS EXERCISES.

1. AB and AC are unequal sides of a triangle ABC; AX is the median through A, AP bisects the angle BAC, and AD is the perpendicular from A to BC. Prove that AP is intermediate in position and magnitude to AX and AD.

2. In a triangle if a perpendicular is drawn from one extremity of the base to the bisector of the vertical angle, (i) it will make with either of the sides containing the vertical angle an angle equal to half the sum of the angles at the base; (ii) it will make with the base an angle equal to half the difference of the angles at the base.

3. In any triangle the angle contained by the bisector of the vertical angle and the perpendicular from the vertex to the base is equal to half the difference of the angles at the base.

4. Construct a right-angled triangle having given the hypotenuse and the difference of the other sides.

5. Construct a triangle, having given the base, the difference of the angles at the base, and (i) the difference, (ii) the sum of the remaining sides.

6. Construct an isosceles triangle, having given the base and the sum of one of the equal sides and the perpendicular from the vertex to the base.

7. Shew how to divide a given straight line so that the square on one part may be double of the square on the other.

8. ABCD is a parallelogram, and O is any point without the angle BAD or its opposite vertical angle; shew that the triangle OAC is equal to the sum of the triangles OAD, OAB.

If O is within the angle BAD or its opposite vertical angle, shew that the triangle OAC is equal to the difference of the triangles OAD, OAB.

9. The area of a quadrilateral is equal to the area of a triangle having two of its sides equal to the diagonals of the given figure, and the included angle equal to either of the angles between the diagonals.

10. Find the locus of the intersection of the medians of triangles described on a given base and of given area.

11. On the base of a given triangle construct a second triangle equal in area to the first, and having its vertex in a given straight line.

12. ABCD is a parallelogram made of rods connected by hinges. If AB is fixed, find the locus of the middle point of CD.

PART III.

THE CIRCLE.

DEFINITIONS AND FIRST PRINCIPLES.

1. **A circle** is a plane figure contained by a line traced out by a point which moves so that its distance from a certain fixed point is always the same.

The fixed point is called the **centre**, and the bounding line is called the **circumference**.

NOTE. According to this definition the term circle strictly applies to the *figure* contained by the circumference; it is often used however for the circumference itself when no confusion is likely to arise.

2. A **radius** of a circle is a straight line drawn from the centre to the circumference. It follows that all radii of a circle are equal.

3. A **diameter** of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

4. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

It will be proved on page 142 that a diameter divides a circle into two identically equal parts.

5. Circles that have the same centre are said to be **concentric**.

From these definitions we draw the following inferences:

(i) A circle is a *closed* curve; so that if the circumference is crossed by a straight line, this line if produced will cross the circumference at a second point.

(ii) The distance of a point from the centre of a circle is greater or less than the radius according as the point is without or within the circumference.

(iii) A point is outside or inside a circle according as its distance from the centre is greater or less than the radius.

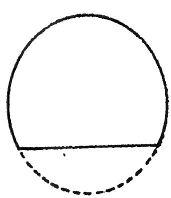
(iv) Circles of equal radii are identically equal. For by superposition of one centre on the other the circumferences must coincide at every point.

(v) Concentric circles of unequal radii cannot intersect, for the distance from the centre of every point on the smaller circle is less than the radius of the larger.

(vi) If the circumferences of two circles have a common point they cannot have the same centre, unless they coincide altogether.

6. An **arc** of a circle is any part of the circumference.

7. A **chord** of a circle is a straight line joining any two points on the circumference.



NOTE. From these definitions it may be seen that a chord of a circle, which does not pass through the centre, divides the circumference into two unequal arcs; of these, the greater is called the **major arc**, and the less the **minor arc**. Thus the major arc is *greater*, and the minor arc *less* than the semi-circumference.

The major and minor arcs, into which a circumference is divided by a chord, are said to be **conjugate** to one another.

SYMMETRY.

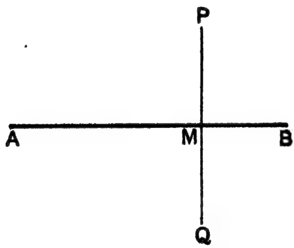
Some elementary properties of circles are easily proved by considerations of symmetry. For convenience the definition given on page 21 is here repeated.

DEFINITION 1. A figure is said to be **symmetrical about a line** when, on being folded about that line, the parts of the figure on each side of it can be brought into coincidence.

The straight line is called an **axis of symmetry.**

That this may be possible, it is clear that the two parts of the figure must have the same size and shape, and must be similarly placed with regard to the axis.

DEFINITION 2. Let AB be a straight line and P a point outside it.



From P draw PM perp. to AB, and produce it to Q, making MQ equal to PM.

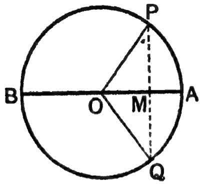
Then if the figure is folded about AB, the point P may be made to coincide with Q, for the $\angle$AMP = the $\angle$AMQ, and MP = MQ.

The points P and Q are said to be **symmetrically opposite** with regard to the axis AB, and each point is said to be the **image** of the other in the axis.

NOTE. A point and its image are equidistant from every point on the axis. See Prob. 14, page 91.

SOME SYMMETRICAL PROPERTIES OF CIRCLES.

I. *A circle is symmetrical about any diameter.*



Let APBQ be a circle of which O is the centre, and AB any diameter.

It is required to prove that the circle is symmetrical about AB.

Proof. Let OP and OQ be two radii making any equal $\angle^s$ AOP, AOQ on opposite sides of OA.

Then if the figure is folded about AB, OP may be made to fall along OQ, since the $\angle$AOP = the $\angle$AOQ.

And thus P will coincide with Q, since OP = OQ.

Thus every point in the arc APB must coincide with some point in the arc AQB; that is, the two parts of the circumference on each side of AB can be made to coincide.

$\therefore$ the circle is symmetrical about the diameter AB.

COROLLARY. If PQ is drawn cutting AB at M, then on folding the figure about AB, since P falls on Q, MP will coincide with MQ,

$\therefore$ MP = MQ;

and the $\angle$OMP will coincide with the $\angle$OMQ,

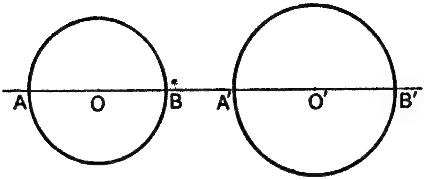
$\therefore$ these angles, being adjacent, are rt. $\angle^s$;

$\therefore$ the points P and Q are symmetrically opposite with regard to AB.

Hence, conversely, *if a circle passes through a given point* P, *it also passes through the symmetrically opposite point with regard to any diameter.*

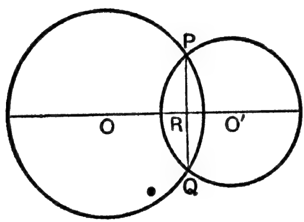
DEFINITION. The straight line passing through the centres of two circles is called the **line of centres.**

II. *Two circles are divided symmetrically by their line of centres.*



Let O, O′ be the centres of two circles, and let the st. line through O, O′ cut the O^{ces} at A, B and A′, B′. Then AB and A′B′ are diameters and therefore axes of symmetry of their respective circles. That is, the line of centres divides each circle symmetrically.

III. *If two circles cut at one point, they must also cut at a second point; and the common chord is bisected at right angles by the line of centres.*



Let the circles whose centres are O, O′ cut at the point P.

Draw PR perp. to OO′, and produce it to Q, so that RQ = RP.

Then P and Q are symmetrically opposite points with regard to the line of centres OO′;

∴ since P is on the O^{ce} of both circles, it follows that Q is also on the O^{ce} of both. [I. Cor.]

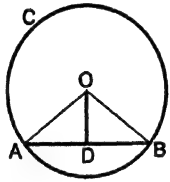
And, by construction, the common chord PQ is bisected at right angles by OO′.

ON CHORDS.

Theorem 31. [Euclid III. 3.]

If a straight line drawn from the centre of a circle bisects a chord which does not pass through the centre, it cuts the chord at right angles.

Conversely, if it cuts the chord at right angles, it bisects it.



Let ABC be a circle whose centre is O; and let OD bisect a chord AB which does not pass through the centre.

It is required to prove that OD *is perp. to* AB.

Join OA, OB.

Proof. Then in the $\triangle^s$ ADO, BDO,

because $\begin{cases} \text{AD} = \text{BD, by hypothesis,} \\ \text{OD is common,} \\ \text{and OA} = \text{OB, being radii of the circle;} \end{cases}$

∴ the $\angle$ ADO = the $\angle$ BDO; *Theor.* 7.

and these are adjacent angles,

∴ OD is perp. to AB. Q.E.D.

Conversely. Let OD be perp. to the chord AB.

It is required to prove that OD *bisects* AB.

Proof. In the $\triangle^s$ ODA, ODB,

because $\begin{cases} \text{the } \angle^s \text{ ODA, ODB are } \textit{right angles,} \\ \text{the hypotenuse OA} = \text{the hypotenuse OB,} \\ \text{and OD is common;} \end{cases}$

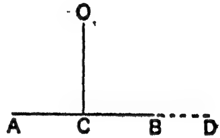
∴ DA = DB; *Theor.* 18.

that is, OD bisects AB at D. Q.E.D.

COROLLARY 1. *The straight line which bisects a chord at right angles passes through the centre.*

COROLLARY 2. *A straight line cannot meet a circle at more than two points.*

For suppose a st. line meets a circle whose centre is O at the points A and B.



Draw OC perp. to AB.

Then AC = CB.

Now if the circle were to cut AB in a third point D, AC would also be equal to CD, which is impossible.

COROLLARY 3. *A chord of a circle lies wholly within it.*

EXERCISES.

(*Numerical and Graphical.*)

1. In the figure of Theorem 31, if AB = 8 cm., and OD = 3 cm., find OB. Draw the figure, and verify your result by measurement.

2. Calculate the length of a chord which stands at a distance 5″ from the centre of a circle whose radius is 13″.

3. In a circle of 1″ radius draw two chords 1·6″ and 1·2″ in length. Calculate and measure the distance of each from the centre.

4. Draw a circle whose diameter is 8·0 cm. and place in it a chord 6·0 cm. in length. Calculate to the nearest millimetre the distance of the chord from the centre; and verify your result by measurement.

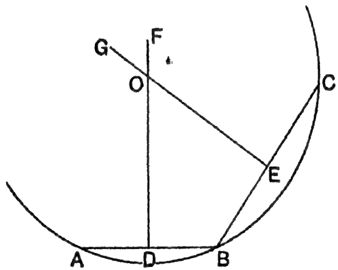
5. Find the distance from the centre of a chord 5 ft. 10 in. in length in a circle whose diameter is 2 yds. 2 in. Verify the result graphically by drawing a figure in which 1 cm. represents 10″.

6. AB is a chord 2·4″ long in a circle whose centre is O and whose radius is 1·3″; find the area of the triangle OAB in square inches.

7. Two points P and Q are 3″ apart. Draw a circle with radius 1·7″ to pass through P and Q. Calculate the distance of its centre from the chord PQ, and verify by measurement.

THEOREM 32.

One circle, and only one, can pass through any three points not in the same straight line.



Let A, B, C be three points not in the same straight line.

It is required to prove that one circle, and only one, can pass through A, B, and C.

Join AB, BC.

Let AB and BC be bisected at right angles by the lines DF, EG.

Then since AB and BC are not in the same st. line, DF and EG are not par^l.

Let DF and EG meet in O.

Proof. Because DF bisects AB at right angles,

∴ every point on DF is equidistant from A and B.

Prob. 14.

Similarly every point on EG is equidistant from B and C.

∴ O, the only point common to DF and EG, is equidistant from A, B, and C;

and there is no other point equidistant from A, B, and C.

∴ a circle having its centre at O and radius OA will pass through B and C; and this is the only circle which will pass through the three given points. Q.E.D.

COROLLARY 1. *The size and position of a circle are fully determined if it is known to pass through three given points;* for then the position of the centre and length of the radius can be found.

COROLLARY 2. *Two circles cannot cut one another in more than two points without coinciding entirely;* for if they cut at three points they would have the same centre and radius.

HYPOTHETICAL CONSTRUCTION. *From Theorem 32 it appears that we may suppose a circle to be drawn through any three points not in the same straight line.*

For example, a circle can be assumed to pass through the vertices of any triangle.

DEFINITION. The circle which passes through the vertices of a triangle is called its **circum-circle**, and is said to be **circumscribed** about the triangle. The centre of the circle is called the **circum-centre** of the triangle, and the radius is called the **circum-radius**.

EXERCISES ON THEOREMS 31 AND 32.

(*Theoretical.*)

1. The parts of a straight line intercepted between the circumferences of two concentric circles are equal.

2. Two circles, whose centres are at A and B, intersect at C, D; and M is the middle point of the common chord. Shew that AM and BM are in the same straight line.

Hence prove that the line of centres bisects the common chord at right angles.

3. AB, AC are two equal chords of a circle; shew that the straight line which bisects the angle BAC passes through the centre.

4. *Find the locus of the centres of all circles which pass through two given points.*

5. *Describe a circle that shall pass through two given points and have its centre in a given straight line.*

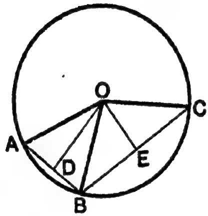
When is this impossible?

6. *Describe a circle of given radius to pass through two given points.*

When is this impossible?

* THEOREM 33. [Euclid III. 9.]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and O a point within it from which more than two equal st. lines are drawn to the O^{ce}, namely OA, OB, OC.

It is required to prove that O *is the centre of the circle* ABC.

Join AB, BC.

Let D and E be the middle points of AB and BC respectively.

Join OD, OE.

Proof. In the $\triangle^s$ ODA, ODB,

$$\text{because}\begin{cases} DA = DB, \\ DO \text{ is common,} \\ \text{and } OA = OB, \text{ by hypothesis;} \end{cases}$$

$\therefore$ the $\angle ODA =$ the $\angle ODB$; *Theor.* 7.

$\therefore$ these angles, being adjacent, are rt. $\angle^s$.

Hence DO bisects the chord AB at right angles, and therefore passes through the centre. *Theor.* 31, *Cor.* 1.

Similarly it may be shewn that EO passes through the centre.

$\therefore$ O, which is the only point common to DO and EO, must be the centre. Q.E.D.

EXERCISES ON CHORDS.

(*Numerical and Graphical.*)

1. AB and BC are lines at right angles, and their lengths are 1·6″ and 3·0″ respectively. Draw the circle through the points A, B, and C; find the length of its radius, and verify your result by measurement.

2. Draw a circle in which a chord 6 cm. in length stands at a distance of 3 cm. from the centre.

Calculate (to the nearest millimetre) the length of the radius, and verify your result by measurement.

3. Draw a circle on a diameter of 8 cm., and place in it a chord equal to the radius.

Calculate (to the nearest millimetre) the distance of the chord from the centre, and verify by measurement.

4. Two circles, whose radii are respectively 26 inches and 25 inches, intersect at two points which are 4 feet apart. Find the distance between their centres.

Draw the figure (scale 1 cm. to 10″), and verify your result by measurement.

5. Two parallel chords of a circle whose diameter is 13″ are respectively 5″ and 12″ in length: shew that the distance between them is either 8·5″ or 3·5″.

6. Two parallel chords of a circle on the same side of the centre are 6 cm. and 8 cm. in length respectively, and the perpendicular distance between them is 1 cm. Calculate and measure the radius.

7. Shew on squared paper that if a circle has its centre at *any* point on the x-axis and passes through the point (6, 5), it also passes through the point (6, −5). [See page 132.]

(*Theoretical.*)

8. The line joining the middle points of two parallel chords of a circle passes through the centre.

9. *Find the locus of the middle points of parallel chords in a circle.*

10. Two intersecting chords of a circle cannot bisect each other unless each is a diameter.

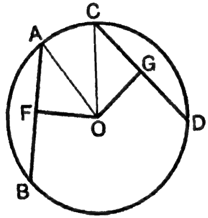
11. If a parallelogram can be inscribed in a circle, the point of intersection of its diagonals must be at the centre of the circle.

12. Shew that rectangles are the only parallelograms that can be inscribed in a circle.

THEOREM 34. [Euclid III. 14.]

Equal chords of a circle are equidistant from the centre.

Conversely, chords which are equidistant from the centre are equal.



Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O.

First. Let AB = CD.

It is required to prove that AB *and* CD *are equidistant from* O.

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

∴ OF bisects AB; *Theor.* 31.

∴ AF is half of AB.

Similarly CG is half of CD.

But, by hypothesis, AB = CD,

∴ AF = CG.

Now in the $\triangle^s$ OFA, OGC,

because $\begin{cases} \text{the } \angle^s \text{ OFA, OGC are } \textit{right angles}, \\ \text{the hypotenuse OA} = \text{the hypotenuse OC}, \\ \text{and AF} = \text{CG}; \end{cases}$

∴ the triangles are equal in all respects; *Theor.* 18.

so that OF = OG;

that is, AB and CD are equidistant from O.

Q.E.D.

Conversely. Let $OF = OG$.

It is required to prove that $AB = CD$.

Proof. As before it may be shewn that AF is half of AB, and CG half of CD.

Then in the $\triangle^s$ OFA, OGC,

because $\begin{cases} \text{the } \angle^s \text{ OFA, OGC are } \textit{right angles}, \\ \text{the hypotenuse OA} = \text{the hypotenuse OC}, \\ \text{and OF} = \text{OG}; \end{cases}$

$\therefore$ $AF = CG$; *Theor.* 18.

$\therefore$ the doubles of these are equal;

that is, $AB = CD$.

Q.E.D.

EXERCISES.

(Theoretical.)

1. *Find the locus of the middle points of equal chords of a circle.*

2. If two chords of a circle cut one another, and make equal angles with the straight line which joins their point of intersection to the centre, they are equal.

3. If two equal chords of a circle intersect, shew that the segments of the one are equal respectively to the segments of the other.

4. In a given circle draw a chord which shall be equal to one given straight line (not greater than the diameter) and parallel to another.

5. PQ is a fixed chord in a circle, and AB is any diameter: shew that the sum or difference of the perpendiculars let fall from A and B on PQ is constant, that is, the same for all positions of AB.

[See Ex. 9, p. 65.]

(Graphical.)

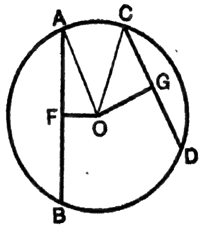
6. In a circle of radius 4·1 cm. any number of chords are drawn each 1·8 cm. in length. Shew that the middle points of these chords all lie on a circle. Calculate and measure the length of its radius, and draw the circle.

7. The centres of two circles are 4″ apart, their common chord is 2·4″ in length, and the radius of the larger circle is 3·7″. Give a construction for finding the points of intersection of the two circles, and find the radius of the smaller circle.

THEOREM 35. [Euclid III. 15.]

Of any two chords of a circle, that which is nearer to the centre is greater than one more remote.

Conversely, the greater of two chords is nearer to the centre than the less.



Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O.

It is required to prove that

(i) *if* OF *is less than* OG, *then* AB *is greater than* CD:

(ii) *if* AB *is greater than* CD, *then* OF *is less than* OG.

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

$\therefore$ OF bisects AB;

$\therefore$ AF is half of AB:

Similarly CG is half of CD.

Now OA = OC,

$\therefore$ the sq. on OA = the sq. on OC.

But since the $\angle$ OFA is a rt. angle,

$\therefore$ the sq. on OA = the sqq. on OF, FA.

Similarly the sq. on OC = the sqq. on OG, GC.

$\therefore$ the sqq. on OF, FA = the sqq. on OG, GC.

(i) Hence if OF is given less than OG;
the sq. on OF is less than the sq. on OG.
∴ the sq. on FA is greater than the sq. on GC;
∴ FA is greater than GC:
∴ AB is greater than CD.

(ii) But if AB is given greater than CD,
that is, if FA is greater than GC;
then the sq. on FA is greater than the sq. on GC.
∴ the sq. on OF is less than the sq. on OG;
∴ OF is less than OG. Q.E.D.

COROLLARY. *The greatest chord in a circle is a diameter.*

EXERCISES.

(Miscellaneous.)

1. Through a given point within a circle draw the least possible chord.

2. Draw a triangle ABC in which $a=3{\cdot}5''$, $b=1{\cdot}2''$, $c=3{\cdot}7''$. Through the ends of the side a draw a circle with its centre on the side c. Calculate and measure the radius.

3. Draw the circum-circle of a triangle whose sides are $2{\cdot}6''$, $2{\cdot}8''$, and $3{\cdot}0''$. Measure its radius.

4. AB is a fixed chord of a circle, and XY any other chord having its middle point Z on AB; what is the greatest, and what the least length that XY may have?

Shew that XY increases, as Z approaches the middle point of AB.

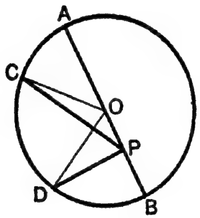
5. Shew on squared paper that a circle whose centre is at the origin, and whose radius is $3{\cdot}0''$, passes through the points $(2{\cdot}4'', 1{\cdot}8'')$, $(1{\cdot}8'', 2{\cdot}4'')$.

Find (i) the length of the chord joining these points, (ii) the coordinates of its middle point, (iii) its perpendicular distance from the origin.

*Theorem 36. [Euclid III. 7.]

If from any internal point, not the centre, straight lines are drawn to the circumference of a circle, then the greatest is that which passes through the centre, and the least is the remaining part of that diameter.

And of any other two such lines the greater is that which subtends the greater angle at the centre.



Let ACDB be a circle, and from P any internal point, which is not the centre, let PA, PB, PC, PD be drawn to the ○ce, so that PA passes through the centre O, and PB is the remaining part of that diameter. Also let the ∠ POC at the centre subtended by PC be greater than the ∠ POD subtended by PD.

It is required to prove that of these st. lines

(i) PA *is the greatest,*

(ii) PB *is the least,*

(iii) PC *is greater than* PD.

Join OC, OD.

Proof. (i) In the △POC, the sides PO, OC are together greater than PC. *Theor.* 11.

But OC = OA, being radii;

∴ PO, OA are together greater than PC;

that is, PA is greater than PC.

Similarly PA may be shewn to be greater than any other st. line drawn from P to the ○ce;

∴ PA is the greatest of all such lines.

(ii) In the $\triangle$ OPD, the sides OP, PD are together greater than OD.

But OD = OB, being radii;

$\therefore$ OP, PD are together greater than OB.

Take away the common part OP;

then PD is greater than PB.

Similarly any other st. line drawn from P to the $\bigcirc^{ce}$ may be shewn to be greater than PB;

$\therefore$ PB is the least of all such lines.

(iii) In the $\triangle^s$ POC, POD,

because $\begin{cases} \text{PO is common,} \\ \text{OC = OD, being radii,} \\ \text{but the } \angle \text{ POC is greater than the } \angle \text{ POD;} \end{cases}$

$\therefore$ PC is greater than PD. *Theor.* 19.

Q.E.D.

EXERCISES.

(*Miscellaneous.*)

1. *All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.*

2. If two circles which intersect are cut by a straight line parallel to the common chord, shew that the parts of it intercepted between the circumferences are equal.

3. If two circles cut one another, any two parallel straight lines drawn through the points of intersection to cut the circles are equal.

4. If two circles cut one another, any two straight lines drawn through a point of section, making equal angles with the common chord, and terminated by the circumferences, are equal.

5. Two circles of diameters 74 and 40 inches respectively have a common chord 2 feet in length: find the distance between their centres.

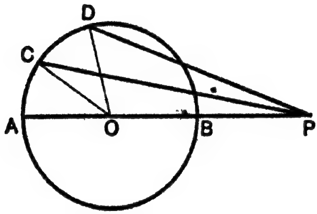
Draw the figure (1 cm. to represent 10") and verify your result by measurement.

6. Draw two circles of radii 1·0" and 1·7", and with their centres 2·1" apart. Find by calculation, and by measurement, the length of the common chord, and its distance from the two centres.

*THEOREM 37. [Euclid III. 8.]

If from any external point straight lines are drawn to the circumference of a circle, the greatest is that which passes through the centre, and the least is that which when produced passes through the centre.

And of any other two such lines, the greater is that which subtends the greater angle at the centre.



Let ACDB be a circle, and from any external point P let the lines PBA, PC, PD be drawn to the $\odot^{ce}$, so that PBA passes through the centre O, and so that the $\angle$ POC subtended by PC at the centre is greater than the $\angle$ POD subtended by PD.

It is required to prove that of these st. lines

(i) PA *is the greatest,*
(ii) PB *is the least,*
(iii) PC *is greater than* PD.

Join OC, OD.

Proof. (i) In the $\triangle$ POC, the sides PO, OC are together greater than PC.

But OC = OA, being radii;

$\therefore$ PO, OA are together greater than PC;

that is, PA is greater than PC.

Similarly PA may be shewn to be greater than any other st. line drawn from P to the $\odot^{ce}$;

that is, PA is the greatest of all such lines.

(ii) In the $\triangle$ POD, the sides PD, DO are together greater than PO.

But OD = OB, being radii;

$\therefore$ the remainder PD is greater than the remainder PB.

Similarly any other st. line drawn from P to the O^{ce} may be shewn to be greater than PB;

that is, PB is the least of all such lines.

(iii) In the $\triangle^s$ POC, POD,

because $\begin{cases} \text{PO is common,} \\ \text{OC = OD, being radii;} \\ \text{but the } \angle \text{ POC is greater than the } \angle \text{ POD;} \end{cases}$

$\therefore$ PC is greater than PD. *Theor.* 19.

Q.E.D.

EXERCISES.

(*Miscellaneous.*)

1. Find the greatest and least straight lines which have one extremity on each of two given circles which do not intersect.

2. If from any point on the circumference of a circle straight lines are drawn to the circumference, the greatest is that which passes through the centre; and of any two such lines the greater is that which subtends the greater angle at the centre.

3. Of all straight lines drawn through a point of intersection of two circles, and terminated by the circumferences, the greatest is that which is parallel to the line of centres.

4. Draw on squared paper any two circles which have their centres on the x-axis, and cut at the point (8, −11). Find the coordinates of their other point of intersection.

5. Draw on squared paper two circles with centres at the points (15, 0) and (−6, 0) respectively, and cutting at the point (0, 8). Find the lengths of their radii, and the coordinates of their other point of intersection.

6. Draw an isosceles triangle OAB with an angle of 80° at its vertex O. With centre O and radius OA draw a circle, and on its circumference take any number of points P, Q, R, ... on the same side of AB as the centre. Measure the angles subtended by the chord AB at the points P, Q, R, ... Repeat the same exercise with any other given angle at O. What inference do you draw?

ON ANGLES IN SEGMENTS, AND ANGLES AT THE CENTRES AND CIRCUMFERENCES OF CIRCLES.

THEOREM 38. [Euclid III. 20.]

The angle at the centre of a circle is double of an angle at the circumference standing on the same arc.

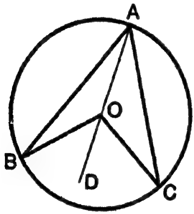


Fig. 1.

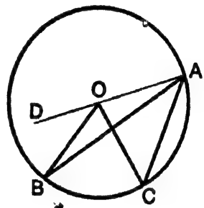


Fig. 2.

Let ABC be a circle, of which O is the centre; and let BOC be the angle at the centre, and BAC an angle at the $\bigcirc^{ce}$, standing on the same arc BC.

It is required to prove that the ∠ BOC *is twice the* ∠ BAC.

Join AO, and produce it to D.

Proof. In the △OAB, because OB = OA,

∴ the ∠OAB = the ∠OBA.

∴ the sum of the ∠^{s} OAB, OBA = twice the ∠OAB.

But the ext. ∠BOD = the sum of the ∠^{s} OAB, OBA;

∴ the ∠BOD = twice the ∠OAB.

Similarly the ∠DOC = twice the ∠OAC.

∴, adding these results in Fig. 1, and taking the difference in Fig. 2, it follows in each case that

the ∠BOC = twice the ∠BAC. Q.E.D.

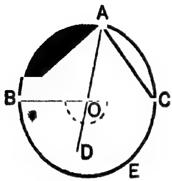


Fig. 3.

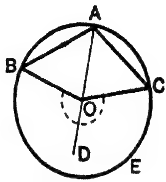
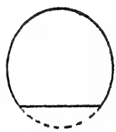


Fig. 4.

Obs. If the arc BEC, on which the angles stand, is a semi-circumference, as in Fig. 3, the $\angle$ BOC at the centre is a *straight angle*; and if the arc BEC is greater than a semi-circumference, as in Fig. 4, the $\angle$ BOC at the centre is *reflex*. But the proof for Fig. 1 applies without change to both these cases, shewing that whether the given arc is greater than, equal to, or less than a semi-circumference,

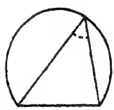
the $\angle$ BOC = *twice the* $\angle$ BAC, *on the same arc* BEC.

DEFINITIONS.



A **segment** of a circle is the figure bounded by a chord and one of the two arcs into which the chord divides the circumference.

NOTE. The chord of a segment is sometimes called its base.



An **angle in a segment** is one formed by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.

We have seen in Theorem 32 that a circle may be drawn through *any three* points not in a straight line. But it is only under certain conditions that a circle can be drawn through more than three points.

DEFINITION. If four or more points are so placed that a circle may be drawn through them, they are said to be **concyclic**.

THEOREM 39. [Euclid III. 21.]

Angles in the same segment of a circle are equal.

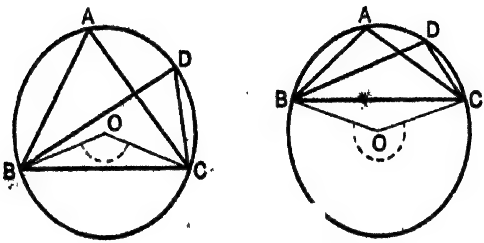


Fig. 1. Fig. 2.

Let BAC, BDC be angles in the same segment BADC of a circle, whose centre is O.

It is required to prove that the ∠ BAC = *the* ∠ BDC.

Join BO, OC.

Proof. Because the ∠ BOC is at the centre, and the ∠ BAC at the ⊙ce, standing on the same arc BC,

∴ the ∠ BOC = twice the ∠ BAC. *Theor.* 38.

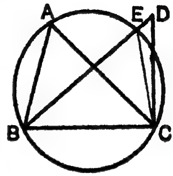
Similarly the ∠ BOC = twice the ∠ BDC.

∴ the ∠ BAC = the ∠ BDC. Q.E.D.

NOTE. The given segment may be greater than a semicircle as in Fig. 1, or less than a semicircle as in Fig. 2: in the latter case the angle BOC will be reflex. But by virtue of the extension of Theorem 38, given on the preceding page, the above proof applies equally to both figures.

CONVERSE OF THEOREM 39.

Equal angles standing on the same base, and on the same side of it, have their vertices on an arc of a circle, of which the given base is the chord.



Let BAC, BDC be two equal angles standing on the same base BC, and on the same side of it.

It is required to prove that A *and* D *lie on an arc of a circle having* BC *as its chord.*

Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts BD or BD produced at the point E.

Join EC.

Proof. Then the $\angle$BAC = the $\angle$BEC in the same segment.

But, by hypothesis, the $\angle$BAC = the $\angle$BDC;

$\therefore$ the $\angle$BEC = the $\angle$BDC;

which is impossible unless E coincides with D;

$\therefore$ the circle through B, A, C must pass through D.

COROLLARY. *The locus of the vertices of triangles drawn on the same side of a given base, and with equal vertical angles is an arc of a circle.*

EXERCISES ON THEOREM 39.

1. In Fig. 1, if the angle BDC is 74°, find the number of degrees in each of the angles BAC, BOC, OBC.

2. In Fig. 2, let BD and CA intersect at X. If the angle DXC = 40°, and the angle XCD = 25°, find the number of degrees in the angle BAC and in the reflex angle BOC.

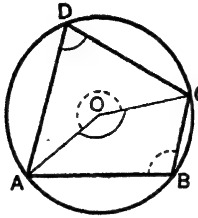
3. In Fig. 1, if the angles CBD, BCD are respectively 48° and 82°, find the number of degrees in the angles BAC, OBD, OCD.

4. Shew that in Fig. 2 the angle OBC is always less than the angle BAC by a right angle.

[For further Exercises on Theorem 39 see page 170.]

THEOREM 40. [Euclid III. 22.]

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.



Let ABCD be a quadrilateral inscribed in the ⊙ABC.

It is required to prove that

(i) *the* ∠ˢ ADC, ABC *together* = *two rt. angles.*

(ii) *the* ∠ˢ BAD, BCD *together* = *two rt. angles.*

Suppose O is the centre of the circle.

Join OA, OC.

Proof. Since the ∠ADC at the Oᶜᵉ = half the ∠AOC at the centre, standing on the same arc ABC;
and the ∠ABC at the Oᶜᵉ = half the reflex ∠AOC at the centre, standing on the same arc ADC;
∴ the ∠ˢ ADC, ABC together = half the sum of the ∠AOC and the reflex ∠AOC.

But these angles make up four rt. angles.

∴ the ∠ˢ ADC, ABC together = two rt. angles.

Similarly the ∠ˢ BAD, BCD together = two rt. angles.

Q.E.D.

NOTE. The results of Theorems 39 and 40 should be carefully compared.

From Theorem 39 we learn that angles in the *same* segment are *equal.*

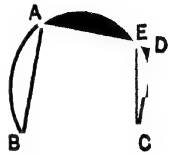
From Theorem 40 we learn that angles in *conjugate* segments are *supplementary.*

DEFINITION. A quadrilateral is called **cyclic** when a circle can be drawn through its four vertices.

CONVERSE OF THEOREM 40.

If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

Let ABCD be a quadrilateral in which the opposite angles at B and D are supplementary.



It is required to prove that the points A, B, C, D *are concyclic.*

Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts AD or AD produced in the point E.

Join EC.

Proof. Then since ABCE is a cyclic quadrilateral,

∴ the ∠ AEC is the supplement of the ∠ ABC.

But, by hypothesis, the ∠ ADC is the supplement of the ∠ABC;

∴ the ∠ AEC = the ∠ ADC;

which is impossible unless E coincides with D.

∴ the circle which passes through A, B, C must pass through D:

that is, A, B, C, D are concyclic. Q.E.D.

EXERCISES ON THEOREM 40.

1. In a circle of 1·6″ radius inscribe a quadrilateral ABCD, making the angle ABC equal to 126°. Measure the remaining angles, and hence verify in this case that opposite angles are supplementary.

2. Prove Theorem 40 by the aid of Theorems 39 and 16, after first joining the opposite vertices of the quadrilateral.

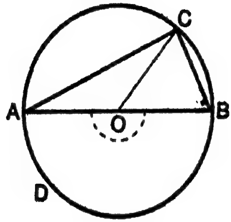
3. If a circle can be described about a parallelogram, the parallelogram must be rectangular.

4. ABC is an isosceles triangle, and XY is drawn parallel to the base BC cutting the sides in X and Y: shew that the four points B, C, X, Y lie on a circle.

5. *If one side of a cyclic quadrilateral is produced, the exterior angle is equal to the opposite interior angle of the quadrilateral.*

THEOREM 41. [Euclid III. 31.]

The angle in a semi-circle is a right angle.



Let ADB be a circle of which AB is a diameter and O the centre; and let C be any point on the semi-circumference ACB.

It is required to prove that the ∠ ACB *is a rt. angle.*

1st Proof. The ∠ ACB at the O^{ce} is half the *straight angle* AOB at the centre, standing on the same arc ADB;

and a *straight angle* = two rt. angles:

∴ the ∠ ACB is a rt. angle. Q.E.D.

2nd Proof. Join OC.

Then because OA = OC,

∴ the ∠ OCA = the ∠ OAC. *Theor.* 5.

And because OB = OC,

∴ the ∠ OCB = the ∠ OBC.

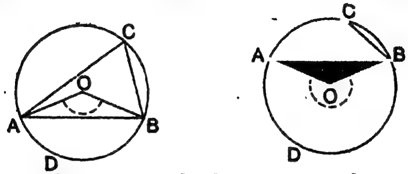
∴ the whole ∠ ACB = the ∠ OAC + the ∠ OBC.

But the three angles of the △ ACB together = two rt. angles;

∴ the ∠ ACB = one-half of two rt. angles

= one rt. angle. Q.E.D.

COROLLARY. *The angle in a segment greater than a semi-circle is acute; and the angle in a segment less than a semi-circle is obtuse.*



The ∠ACB at the ⊙^{ce} is half the ∠AOB at the centre, on the same arc ADB.

(i) If the segment ACB is greater than a semi-circle, then ADB is a *minor* arc;

∴ the ∠AOB is *less* than two rt. angles;
∴ the ∠ACB is *less* than one rt. angle.

(ii) If the segment ACB is less than a semi-circle, then ADB is a *major* arc;

∴ the ∠AOB is *greater* than two rt. angles;
∴ the ∠ACB is *greater* than one rt. angle.

EXERCISES ON THEOREM 41.

1. *A circle described on the hypotenuse of a right-angled triangle as diameter, passes through the opposite angular point.*

2. Two circles intersect at A and B; and through A two diameters AP, AQ are drawn, one in each circle: shew that the points P, B, Q are collinear.

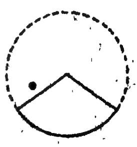
3. A circle is described on one of the equal sides of an isosceles triangle as diameter. Shew that it passes through the middle point of the base.

4. Circles described on any two sides of a triangle as diameters intersect on the third side, or the third side produced.

5. A straight rod of given length slides between two straight rulers placed at right angles to one another; find the locus of its middle point.

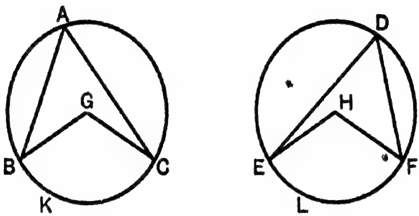
6. *Find the locus of the middle points of chords of a circle drawn through a fixed point. Distinguish between the cases when the given point is within, on, or without the circumference.*

DEFINITION. A sector of a circle is a figure bounded by two radii and the arc intercepted between them.



THEOREM 42. [Euclid III. 26.]

In equal circles, arcs which subtend equal angles, either at the centres or at the circumferences, are equal.



Let ABC, DEF be equal circles, and let the $\angle$ BGC = the $\angle$ EHF at the centres; and consequently

the $\angle$ BAC = the $\angle$ EDF at the $\odot^{ces}$. *Theor.* 38.

It is required to prove that the arc BKC = *the arc* ELF.

Proof. Apply the $\odot$ ABC to the $\odot$ DEF, so that the centre G falls on the centre H, and GB falls along HE.

Then because the $\angle$ BGC = the $\angle$ EHF,

$\therefore$ GC will fall along HF.

And because the circles have equal radii, B will fall on E, and C on F, and the circumferences of the circles will coincide entirely.

$\therefore$ the arc BKC must coincide with the arc ELF;

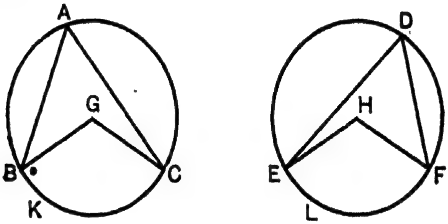
$\therefore$ the arc BKC = the arc ELF. Q.E.D.

COROLLARY. *In equal circles sectors which have equal angles are equal.*

Obs. It is clear that any theorem relating to arcs, angles, and chords in *equal* circles must also be true in *the same* circle.

THEOREM 43. [Euclid III. 27.]

In equal circles angles, either at the centres or at the circumferences, which stand on equal arcs are equal.



Let ABC, DEF be equal circles;
and let the arc BKC = the arc ELF.

It is required to prove that
the $\angle$ BGC = *the* $\angle$ EHF *at the centres;*
also the $\angle$ BAC = *the* $\angle$ EDF *at the* $\bigcirc^{ces}$.

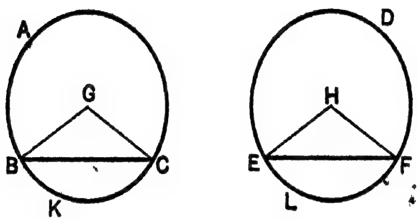
Proof. Apply the $\odot$ ABC to the $\odot$ DEF, so that the centre G falls on the centre H, and GB falls along HE.

Then because the circles have equal radii,
$\therefore$ B falls on E, and the two $\bigcirc^{ces}$ coincide entirely.
And, by hypothesis, the arc BKC = the arc ELF,
$\therefore$ C falls on F, and consequently GC on HF;
$\therefore$ the $\angle$ BGC = the $\angle$ EHF.

And since the $\angle$ BAC at the $\bigcirc^{ce}$ = half the $\angle$ BGC at the centre;
and likewise the $\angle$ EDF = half the $\angle$ EHF;
$\therefore$ the $\angle$ BAC = the $\angle$ EDF. Q.E.D.

Theorem 44. [Euclid III. 28.]

In equal circles, arcs which are cut off by equal chords are equal, the major arc equal to the major arc, and the minor to the minor.



Let ABC, DEF be equal circles whose centres are G and H,
and let the chord BC = the chord EF.

It is required to prove that

the major arc BAC = *the major arc* EDF,

and *the minor arc* BKC = *the minor arc* ELF.

Join BG, GC, EH, HF.

Proof. In the $\triangle$s BGC, EHF,

because { BG = EH, being radii of equal circles,
GC = HF, for the same reason,
and BC = EF, by hypothesis;

$\therefore$ the $\angle$ BGC = the $\angle$ EHF; *Theor.* 7.

$\therefore$ the arc BKC = the arc ELF; *Theor.* 42.

and these are the minor arcs.

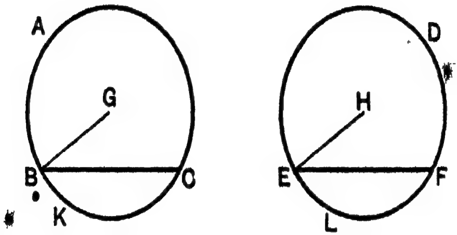
But the whole $\odot^{ce}$ ABKC = the whole $\odot^{ce}$ DELF;

$\therefore$ the remaining arc BAC = the remaining arc EDF:

and these are the major arcs. Q.E.D.

THEOREM 45. [EUCLID III. 29.]

In equal circles chords which cut off equal arcs are equal.



Let ABC, DEF be equal circles whose centres are G and H; and let the arc BKC = the arc ELF.

It is required to prove that the chord BC = *the chord* EF.

Join BG, EH.

Proof. Apply the ⊙ABC to the ⊙DEF, so that G falls on H and GB along HE.

Then because the circles have equal radii,

∴ B falls on E, and the ⊙^ces coincide entirely.

And because the arc BKC = the arc ELF,

∴ C falls on F.

∴ the chord BC coincides with the chord EF;

∴ the chord BC = the chord EF. Q.E.D.

EXERCISES ON ANGLES IN A CIRCLE.

1. P is any point on the arc of a segment of which AB is the chord. Shew that the sum of the angles PAB, PBA is constant.

2. PQ and RS are two chords of a circle intersecting at X: prove that the triangles PXS, RXQ are equiangular to one another.

3. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that PQ subtends a constant angle at B.

4. Two circles intersect at A and B; and through A any two straight lines PAQ, XAY are drawn terminated by the circumferences; shew that the arcs PX, QY subtend equal angles at B.

5. P is any point on the arc of a segment whose chord is AB: and the angles PAB, PBA are bisected by straight lines which intersect at O. Find the locus of the point O.

6. *If two chords intersect within a circle, they form an angle equal to that at the centre, subtended by half the sum of the arcs they cut off.*

7. *If two chords intersect without a circle, they form an angle equal to that at the centre subtended by half the difference of the arcs they cut off.*

8. The sum of the arcs cut off by two chords of a circle at right angles to one another is equal to the semi-circumference.

9. *If* AB *is a fixed chord of a circle and* P *any point on one of the arcs cut off by it, then the bisector of the angle* APB *cuts the conjugate arc in the same point for all positions of* P.

10. AB, AC are any two chords of a circle; and P, Q are the middle points of the minor arcs cut off by them; if PQ is joined, cutting AB in X and AC in Y, shew that AX = AY.

11. A triangle ABC is inscribed in a circle, and the bisectors of the angles meet the circumference at X, Y, Z. Shew that the angles of the triangle XYZ are respectively

$$90° - \frac{A}{2}, \quad 90° - \frac{B}{2}, \quad 90° - \frac{C}{2}.$$

12. Two circles intersect at A and B; and through these points lines are drawn from any point P on the circumference of one of the circles: shew that when produced they intercept on the other circumference an arc which is constant for all positions of P.

13. The straight lines which join the extremities of parallel chords in a circle (i) towards the same parts, (ii) towards opposite parts, are equal.

14. Through A, a point of intersection of two equal circles, two straight lines PAQ, XAY are drawn: shew that the chord PX is equal to the chord QY.

15. Through the points of intersection of two circles two parallel straight lines are drawn terminated by the circumferences: shew that the straight lines which join their extremities towards the same parts are equal.

16. Two equal circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that BP=BQ.

17. ABC is an isosceles triangle inscribed in a circle, and the bisectors of the base angles meet the circumference at X and Y. Shew that the figure BXAYC must have four of its sides equal.

What relation must subsist among the angles of the triangle ABC, in order that the figure BXAYC may be equilateral?

18. ABCD is a cyclic quadrilateral, and the opposite sides AB, DC are produced to meet at P, and CB, DA to meet at Q: if the circles circumscribed about the triangles PBC, QAB intersect at R, shew that the points P, R, Q are collinear.

19. P, Q, R *are the middle points of the sides of a triangle, and* X *is the foot of the perpendicular let fall from one vertex on the opposite side: shew that the four points* P, Q, R, X *are concyclic.*

[See page 64, Ex. 2: also Prob. 10, p. 83.]

20. *Use the preceding exercise to shew that the middle points of the sides of a triangle and the feet of the perpendiculars let fall from the vertices on the opposite sides, are concyclic.*

21. If a series of triangles are drawn standing on a fixed base, and having a given vertical angle, shew that the bisectors of the vertical angles all pass through a fixed point.

22. ABC is a triangle inscribed in a circle, and E the middle point of the arc subtended by BC on the side remote from A: if through E a diameter ED is drawn, shew that the angle DEA is half the difference of the angles at B and C.

TANGENCY.

Definitions and First Principles.

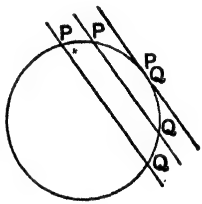
1. A **secant** of a circle is a straight line of indefinite length which cuts the circumference at two points.

2. If a secant moves in such a way that the two points in which it cuts the circle continually approach one another, then in the ultimate position when these two points become one, the secant becomes a **tangent** to the circle, and is said to **touch** it at the point at which the two intersections coincide. This point is called the **point of contact**.

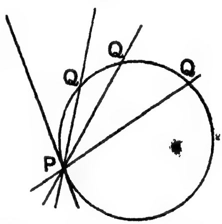
For instance:

(i) Let a secant cut the circle at the points P and Q, and suppose it to recede from the centre, moving always parallel to its original position; then the two points P and Q will clearly approach one another and finally coincide.

In the ultimate position when P and Q become one point, the straight line becomes a tangent to the circle at that point.



(ii) Let a secant cut the circle at the points P and Q, and suppose it to be turned about the point P so that while P remains fixed, Q moves on the circumference nearer and nearer to P. Then the line PQ in its ultimate position, when Q coincides with P, is a tangent at the point P.



Since a secant can cut a circle at *two* points only, it is clear that a tangent can have only one point in common with the circumference, namely the point of contact, at which two points of section coincide. Hence we may define a tangent as follows:

3. A **tangent** to a circle is a straight line which meets the circumference at one point only; and though produced indefinitely does not cut the circumference.

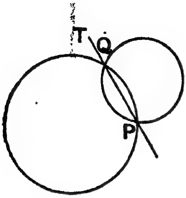


Fig. 1.

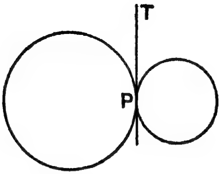


Fig. 2.

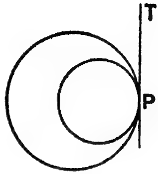


Fig. 3.

4. Let two circles intersect (as in Fig. 1) in the points P and Q, and let one of the circles turn about the point P, which remains fixed, in such a way that Q continually approaches P. Then in the ultimate position, when Q coincides with P (as in Figs. 2 and 3), the circles are said to **touch** one another at P.

Since two circles cannot intersect in more than *two* points, two circles which touch one another cannot have more than *one* point in common, namely the point of contact at which the two points of section coincide. Hence circles are said to **touch** one another when they meet, but do not cut one another.

NOTE. When each of the circles which meet is *outside the other*, as in Fig. 2, they are said to touch one another **externally**, or to have **external contact**: when one of the circles is *within the other*, as in Fig. 3, the first is said to touch the other **internally**, or to have **internal contact** with it.

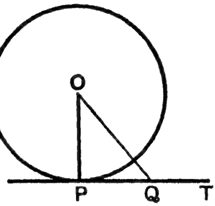
INFERENCE FROM DEFINITIONS 2 AND 4.

If in Fig. 1, TQP is a common chord of two circles one of which is made to turn about P, then when Q is brought into coincidence with P, the line TP passes through two coincident points on each circle, as in Figs. 2 and 3, and therefore becomes a tangent to each circle. Hence

Two circles which touch one another have a common tangent at their point of contact.

THEOREM 46.

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.



Let PT be a tangent at the point P to a circle whose centre is O.

It is required to prove that PT *is perpendicular to the radius* OP.

Proof. Take *any* point Q in PT, and join OQ.

Then since PT is a tangent, every point in it except P is outside the circle.

$\therefore$ OQ is greater than the radius OP.

And this is true for every point Q in PT;

$\therefore$ OP is the shortest distance from O to PT.

Hence OP is perp. to PT. *Theor.* 12, *Cor.* 1.

Q.E.D.

COROLLARY 1. Since there can be only one perpendicular to OP at the point P, it follows that *one and only one tangent can be drawn to a circle at a given point on the circumference.*

COROLLARY 2. Since there can be only one perpendicular to PT at the point P, it follows that *the perpendicular to a tangent at its point of contact passes through the centre.*

COROLLARY 3. Since there can be only one perpendicular from O to the line PT, it follows that *the radius drawn perpendicular to the tangent passes through the point of contact.*

THEOREM 46. [By the Method of Limits.]

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.

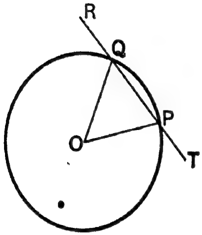


Fig. 1.

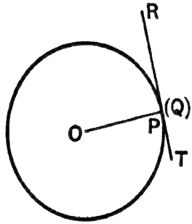


Fig. 2.

Let P be a point on a circle whose centre is O.

It is required to prove that the tangent at P *is perpendicular to the radius* OP.

Let RQPT (Fig. 1) be a secant cutting the circle at Q and P.

Join OQ, OP.

Proof. Because OP = OQ,

∴ the ∠OQP = the ∠OPQ;

∴ the supplements of these angles are equal;

that is, the ∠OQR = the ∠OPT,

and this is true however near Q is to P.

Now let the secant QP be turned about the point P so that Q continually approaches and finally coincides with P; then in the ultimate position,

(i) the secant RT becomes the tangent at P, } Fig. 2,
(ii) OQ coincides with OP; }

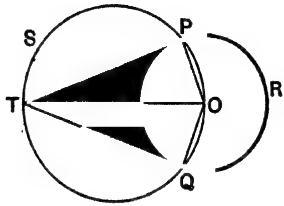
and therefore the equal ∠ˢ OQR, OPT become adjacent,

∴ OP is perp. to RT. Q.E.D.

NOTE. The method of proof employed here is known as the **Method of Limits.**

THEOREM 47.

Two tangents can be drawn to a circle from an external point.



Let PQR be a circle whose centre is O, and let T be an external point.

It is required to prove that there can be two tangents drawn to the circle from T.

Join OT, and let TSO be the circle on OT as diameter.

This circle will cut the ⊙ PQR in two points, since T is without, and O is within, the ⊙ PQR. Let P and Q be these points.

Join TP, TQ; OP, OQ.

Proof. Now each of the ∠ˢ TPO, TQO, being in a semi-circle, is a rt. angle;

∴ TP, TQ are perp. to the radii OP, OQ respectively.

∴ TP, TQ are tangents at P and Q. *Theor.* 46.

Q.E.D.

COROLLARY. *The two tangents to a circle from an external point are equal, and subtend equal angles at the centre.*

For in the △ˢ TPO, TQO,

because the ∠ˢ TPO, TQO are *right angles*, the hypotenuse TO is common, and OP = OQ, being radii;

∴ TP = TQ,

and the ∠ TOP = the ∠ TOQ. *Theor.* 18.

EXERCISES ON THE TANGENT.

(Numerical and Graphical.)

1. Draw two concentric circles with radii 5·0 cm. and 3·0 cm. Draw a series of chords of the former to touch the latter. Calculate and measure their lengths, and account for their being equal.

2. In a circle of radius 1·0″ draw a number of chords each 1·6″ in length. Shew that they all touch a concentric circle, and find its radius.

3. The diameters of two concentric circles are respectively 10·0 cm. and 5·0 cm.: find to the nearest millimetre the length of any chord of the outer circle which touches the inner, and check your work by measurement.

4. In the figure of Theorem 47, if OP=5″, TO=13″, find the length of the tangents from T. Draw the figure (scale 2 cm. to 5″), and measure to the nearest degree the angles subtended at O by the tangents.

5. The tangents from T to a circle whose radius is 0·7″ are each 2·4″ in length. Find the distance of T from the centre of the circle. Draw the figure and check your result graphically.

(Theoretical.)

6. *The centre of any circle which touches two intersecting straight lines must lie on the bisector of the angle between them.*

7. AB and AC are two tangents to a circle whose centre is O; shew that AO bisects the chord of contact BC at right angles.

8. If PQ is joined in the figure of Theorem 47, shew that the angle PTQ is double the angle OPQ.

9. Two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

10. The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

11. *Find the locus of the centres of all circles which touch a given straight line at a given point.*

12. Find the locus of the centres of all circles which touch each of two parallel straight lines.

13. *Find the locus of the centres of all circles which touch each of two intersecting straight lines of unlimited length.*

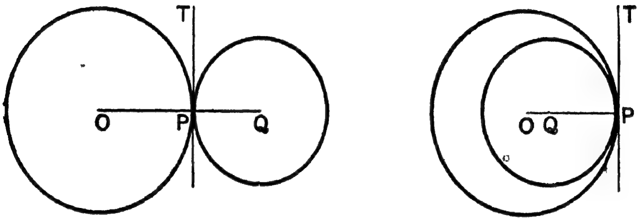
14. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

State and prove the converse theorem.

15. If a quadrilateral is described about a circle, the angles subtended at the centre by any two opposite sides are supplementary.

THEOREM 48.

If two circles touch one another, the centres and the point of contact are in one straight line.



Let two circles whose centres are O and Q touch at the point P.

It is required to prove that O, P, *and* Q *are in one straight line.*

Join OP, QP.

Proof. Since the given circles touch at P, they have a common tangent at that point. *Page* 173.

Suppose PT to touch both circles at P.

Then since OP and QP are radii drawn to the point of contact,

∴ OP and QP are both perp. to PT;

∴ OP and QP are in one st. line. *Theor.* 2.

That is, the points O, P, and Q are in one st. line. Q.E.D.

COROLLARIES. (i) *If two circles touch externally the distance between their centres is equal to the sum of their radii.*

(ii) *If two circles touch internally the distance between their centres is equal to the difference of their radii.*

EXERCISES ON THE CONTACT OF CIRCLES.

(Numerical and Graphical.)

1. From centres 2·6″ apart draw two circles with radii 1·7″ and 0·9″ respectively. Why and where do these circles touch one another?

If circles of the above radii are drawn from centres 0·8″ apart, prove that they touch. How and why does the contact differ from that in the former case?

2. Draw a triangle ABC in which $a=8$ cm., $b=7$ cm., and $c=6$ cm. From A, B, and C as centres draw circles of radii 2·5 cm., 3·5 cm., and 4·5 cm. respectively; and shew that these circles touch in pairs.

3. In the triangle ABC, right-angled at C, $a=8$ cm. and $b=6$ cm.; and from centre A with radius 7 cm. a circle is drawn. What must be the radius of a circle drawn from centre B to touch the first circle?

4. A and B are the centres of two fixed circles which touch internally. If P is the centre of any circle which touches the larger circle internally and the smaller externally, prove that AP + BP is constant.

If the fixed circles have radii 5·0 cm. and 3·0 cm. respectively, verify the general result by taking different positions for P.

5. AB is a line 4″ in length, and C is its middle point. On AB, AC, CB semicircles are described. Shew that if a circle is inscribed in the space enclosed by the three semicircles its radius must be $\frac{2}{3}$″.

(Theoretical.)

6. *A straight line is drawn through the point of contact of two circles whose centres are* A *and* B, *cutting the circumferences at* P *and* Q *respectively; shew that the radii* AP *and* BQ *are parallel.*

7. Two circles touch externally, and through the point of contact a straight line is drawn terminated by the circumferences: shew that the tangents at its extremities are parallel.

8. Find the locus of the centres of all circles

(i) which touch a given circle at a given point;

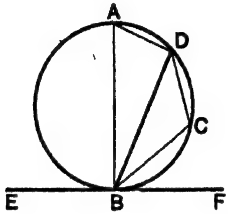
(ii) which are of given radius and touch a given circle.

9. From a given point as centre describe a circle to touch a given circle. How many solutions will there be?

10. Describe a circle of radius a to touch a given circle of radius b at a given point. How many solutions will there be?

THEOREM 49. [Euclid III. 32.]

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.



Let EF touch the ⊙ ABC at B, and let BD be a chord drawn from B, the point of contact.

It is required to prove that

(i) *the* ∠ FBD = *the angle in the alternate segment* BAD:

(ii) *the* ∠ EBD = *the angle in the alternate segment* BCD.

Let BA be the diameter through B, and C any point in the arc of the segment which does not contain A.

Join AD, DC, CB.

Proof. Because the ∠ ADB in a semi-circle is a rt. angle,

∴ the ∠s DBA, BAD together = a rt. angle.

But since EBF is a tangent, and BA a diameter,

∴ the ∠ FBA is a rt. angle.

∴ the ∠ FBA = the ∠s DBA, BAD together.

Take away the common ∠ DBA,

then the ∠ FBD = the ∠ BAD, which is in the alternate segment

Again because ABCD is a cyclic quadrilateral,

∴ the ∠ BCD = the supplement of the ∠ BAD

= the supplement of the ∠ FBD

= the ∠ EBD;

∴ the ∠ EBD = the ∠ BCD, which is in the alternate segment.

Q.E.D.

EXERCISES ON THEOREM 49.

1. In the figure of Theorem 49, if the $\angle FBD = 72°$, write down the values of the $\angle^s$ BAD, BCD, EBD.

2. Use this theorem to shew that tangents to a circle from an external point are equal.

3. Through A, the point of contact of two circles, chords APQ, AXY are drawn: shew that PX and QY are parallel.

Prove this (i) for internal, (ii) for external contact.

4. AB is the common chord of two circles, one of which passes through O, the centre of the other: prove that OA bisects the angle between the common chord and the tangent to the first circle at A.

5. Two circles intersect at A and B; and through P, any point on one of them, straight lines PAC, PBD are drawn to cut the other at C and D: shew that CD is parallel to the tangent at P.

6. If from the point of contact of a tangent to a circle a chord is drawn, the perpendiculars dropped on the tangent and chord from the middle point of either arc cut off by the chord are equal.

EXERCISES ON THE METHOD OF LIMITS.

1. *Prove Theorem 49 by the Method of Limits.*

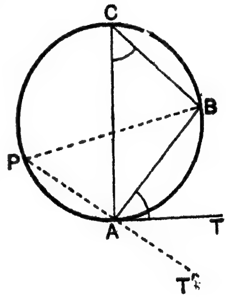
[Let ACB be a segment of a circle of which AB is the chord; and let PAT′ be any *secant* through A. Join PB.

Then the $\angle BCA =$ the $\angle BPA$; *Theor.* 39.

and this is true *however near* P *approaches to* A.

If P moves up to coincidence with A, then the secant PAT′ becomes the tangent AT, and the $\angle BPA$ becomes the $\angle BAT$.

$\therefore$, ultimately, the $\angle BAT =$ the $\angle BCA$, in the alt. segment.]



2. From Theorem 31, prove by the Method of Limits that

The straight line drawn perpendicular to the diameter of a circle at its extremity is a tangent.

3. Deduce Theorem 48 from the property that *the line of centres bisects a common chord at right angles.*

4. Deduce Theorem 49 from Ex. 5, page 163.

5. Deduce Theorem 46 from Theorem 41.

PROBLEMS.

GEOMETRICAL ANALYSIS.

Hitherto the Propositions of this text-book have been arranged **Synthetically**, that is to say, by *building up known results* in order to obtain a *new* result.

But this arrangement, though convincing as an argument, in most cases affords little clue as to the way in which the construction or proof *was discovered*. We therefore draw the student's attention to the following hints.

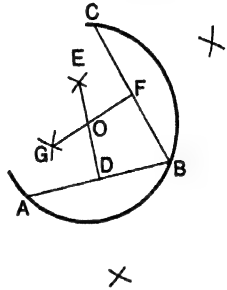
In attempting to solve a problem begin by *assuming* the required result; then by working backwards, trace the consequences of the assumption, and try to ascertain its dependence on some condition or known theorem which suggests the necessary construction. If this attempt is successful, the steps of the argument may in general be re-arranged in reverse order, and the construction and proof presented in a synthetic form.

This unravelling of the conditions of a proposition in order to trace it back to some earlier principle on which it depends, is called **geometrical analysis**: it is the natural way of attacking the harder types of exercises, and it is especially useful in solving problems.

Although the above directions do not amount to a *method*, they often furnish a very effective mode of *searching for a suggestion*. The approach by analysis will be illustrated in some of the following problems. [See Problems 23, 28, 29.]

PROBLEM 20.

Given a circle, or an arc of a circle, to find its centre.



Let ABC be an arc of a circle whose centre is to be found.

Construction. Take two chords AB, BC, and bisect them at right angles by the lines DE, FG, meeting at O. *Prob.* 2.

Then O is the required centre.

Proof. Every point in DE is equidistant from A and B. *Prob.* 14.

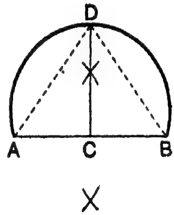
And every point in FG is equidistant from B and C.

∴ O is equidistant from A, B, and C.

∴ O is the centre of the circle ABC. *Theor.* 33

PROBLEM 21.

To bisect a given arc.



Let ADB be the given arc to be bisected.

Construction. Join AB, and bisect it at right angles by CD meeting the arc at D. *Prob.* 2.

Then the arc is bisected at D.

Proof. Join DA, DB.

Then every point on CD is equidistant from A and B; *Prob.* 14.

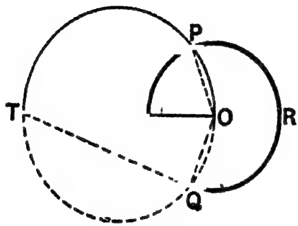
∴ $DA = DB$;

∴ the $\angle DBA =$ the $\angle DAB$; *Theorem* 6.

∴ the arcs, which subtend these angles at the $\bigcirc^{ce}$, are equal; that is, the arc $DA =$ the arc DB.

PROBLEM 22.

To draw a tangent to a circle from a given external point.



Let PQR be the given circle, with its centre at O; and let T be the point from which a tangent is to be drawn.

Construction. Join TO, and on it describe a semi-circle TPO to cut the circle at P.

Join TP.

Then TP is the required tangent.

Proof. Join OP.

Then since the ∠TPO, being in a semi-circle, is a rt. angle,

∴ TP is at right angles to the radius OP.

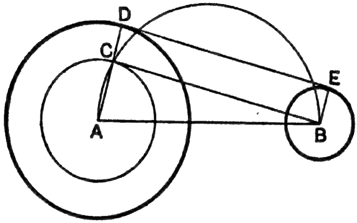
∴ TP is a tangent at P. *Theor.* 46.

Since the semi-circle may be described on either side of TO, a second tangent TQ can be drawn from T, as shewn in the figure.

NOTE. Suppose the point T to approach the given circle, then the angle PTQ gradually increases. When T reaches the circumference, the angle PTQ becomes a *straight angle*, and the two tangents coincide. When T enters the circle, no tangent can be drawn. [See *Obs.* p. 94.]

PROBLEM 23.

To draw a common tangent to two circles.



Let A be the centre of the greater circle, and a its radius; and let B be the centre of the smaller circle, and b its radius.

Analysis. Suppose DE to touch the circles at D and E. Then the radii AD, BE are both perp. to DE, and therefore par[1] to one another.

Now if BC were drawn par[1] to DE, then the fig. DB would be a rectangle, so that $CD = BE = b$.

And if AD, BE are *on the same side of* AB,

then $AC = a - b$, and the $\angle ACB$ is a rt. angle.

These hints enable us to draw BC *first*, and thus lead to the following construction.

Construction. With centre A, and radius equal to the *difference* of the radii of the given circles, describe a circle, and draw BC to touch it.

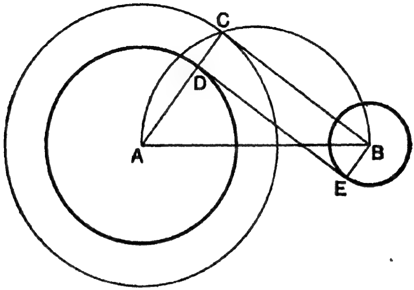
Join AC, and produce it to meet the circle (A) at D.

Through B draw the radius BE par[1] to AD and in the *same sense.* Join DE.

Then DE is a common tangent to the given circles.

Obs. Since two tangents, such as BC, can in general be drawn from B to the circle of construction, this method will furnish two common tangents to the given circles. These are called the **direct common tangents.**

PROBLEM 23. (*Continued.*)



Again, if the circles are *external to one another* two more common tangents may be drawn.

Analysis. In this case we may suppose DE to touch the circles at D and E so that the radii AD, BE fall *on opposite sides of* AB.

Then BC, drawn par[l] to the supposed common tangent DE, would meet AD *produced* at C; and we should now have

AC = AD + DC = $a + b$; and, as before, the ∠ACB is a rt. angle.

Hence the following construction.

Construction. With centre A, and radius equal to the *sum* of the radii of the given circles, describe a circle, and draw BC to touch it.

Then proceed as in the first case, but draw BE in the sense *opposite* to AD.

Obs. As before, two tangents may be drawn from B to the circle of construction; hence two common tangents may be thus drawn to the given circles. These are called the **transverse common tangents.**

[We leave as an exercise to the student the arrangement of the proof in synthetic form.]

EXERCISES ON COMMON TANGENTS.

(Numerical and Graphical.)

1. How many common tangents can be drawn in each of the following cases?

(i) when the given circles intersect;
(ii) when they have external contact;
(iii) when they have internal contact.

Illustrate your answer by drawing two circles of radii 1·4″ and 1·0″ respectively,

(i) with 1·0″ between the centres;
(ii) with 2·4″ between the centres;
(iii) with 0·4″ between the centres;
(iv) with 3·0″ between the centres.

Draw the common tangents in each case, and note where the general construction fails, or is modified.

2. Draw two circles with radii 2·0″ and 0·8″, placing their centres 2·0″ apart. Draw the common tangents, and find their lengths between the points of contact, both by calculation and by measurement.

3. Draw all the common tangents to two circles whose centres are 1·8″ apart and whose radii are 0·6″ and 1·2″ respectively. Calculate and measure the length of the direct common tangents.

4. Two circles of radii 1·7″ and 1·0″ have their centres 2·1″ apart. Draw their common tangents and find their lengths. Also find the length of the common chord. Produce the common chord and shew by measurement that it bisects the common tangents.

5. Draw two circles with radii 1·6″ and 0·8″ and with their centres 3·0″ apart. Draw all their common tangents.

6. Draw the direct common tangents to two equal circles.

(Theoretical.)

7. If the two direct, or the two transverse, common tangents are drawn to two circles, the parts of the tangents intercepted between the points of contact are equal.

8. If four common tangents are drawn to two circles external to one another, shew that the two direct, and also the two transverse, tangents intersect on the line of centres.

9. Two given circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that PQ subtends a right angle at the point A.

On the Construction of Circles.

In order to draw a circle we must know (i) the position of the centre, (ii) the length of the radius.

(i) To find the position of the centre, two conditions are needed, each giving a locus on which the centre must lie; so that the one or more points in which the two loci intersect are possible positions of the required centre, as explained on page 93.

(ii) The position of the centre being thus fixed, the radius is determined if we know (or can find) any point on the circumference.

Hence in order to draw a circle *three* independent data are required.

For example, we may draw a circle if we are given

(i) *three* points on the circumference;

or (ii) *three* tangent lines;

or (iii) one point on the circumference, one tangent, and its point of contact.

It will however often happen that more than one circle can be drawn satisfying three given conditions.

Before attempting the constructions of the next Exercise the student should make himself familiar with the following loci.

(i) *The locus of the centres of circles which pass through two given points.*

(ii) *The locus of the centres of circles which touch a given straight line at a given point.*

(iii) *The locus of the centres of circles which touch a given circle at a given point.*

(iv) *The locus of the centres of circles which touch a given straight line, and have a given radius.*

(v) *The locus of the centres of circles which touch a given circle, and have a given radius.*

(vi) *The locus of the centres of circles which touch two given straight lines.*

EXERCISES.

1. Draw a circle to pass through three given points.

2. If a circle touches a given line PQ at a point A, on what line must its centre lie?

If a circle passes through two given points A and B, on what line must its centre lie?

Hence draw a circle to touch a straight line PQ at the point A, and to pass through another given point B.

3. If a circle touches a given circle whose centre is C at the point A, on what line must its centre lie?

Draw a circle to touch the given circle (C) at the point A, and to pass through a given point B.

4. A point P is 4·5 cm. distant from a straight line AB. Draw two circles of radius 3·2 cm. to pass through P and to touch AB.

5. Given two circles of radius 3·0 cm. and 2·0 cm. respectively, their centres being 6·0 cm. apart; draw a circle of radius 3·5 cm. to touch each of the given circles externally.

How many solutions will there be? What is the radius of the smallest circle that touches each of the given circles externally?

6. If a circle touches two straight lines OA, OB, on what line must its centre lie?

Draw OA, OB, making an angle of 76°, and describe a circle of radius 1·2″ to touch both lines.

7. Given a circle of radius 3·5 cm., with its centre 5·0 cm. from a given straight line AB; draw two circles of radius 2·5 cm. to touch the given circle and the line AB.

8. Devise a construction for drawing a circle to touch each of two parallel straight lines and a transversal.

Shew that two such circles can be drawn, and that they are equal.

9. Describe a circle to touch a given circle, and also to touch a given straight line at a given point. [See page 311.]

10. Describe a circle to touch a given straight line, and to touch a given circle at a given point.

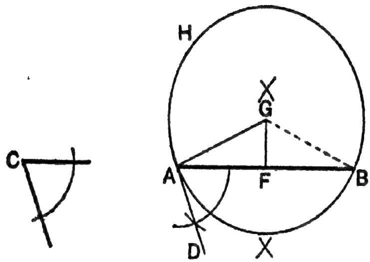
11. Shew how to draw a circle to touch each of three given straight lines of which no two are parallel.

How many such circles can be drawn?

[Further Examples on the Construction of Circles will be found on pp. 246, 311.]

PROBLEM 24.

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.



Let AB be the given st. line, and C the given angle.

It is required to describe on AB *a segment of a circle containing an angle equal to* C.

Construction. At A in BA, make the ∠ BAD equal to the ∠ C.

From A draw AG perp. to AD.

Bisect AB at rt. angles by FG, meeting AG in G. *Prob.* 2

Proof. Join GB.

Now every point in FG is equidistant from A and B; *Prob.* 14.

∴ GA = GB.

With centre G, and radius GA, draw a circle, which must pass through B, and touch AD at A. *Theor.* 46.

Then the segment AHB, alternate to the ∠ BAD, contains an angle equal to C. *Theor.* 49.

NOTE. In the particular case when the given angle is a rt. angle, the segment required will be the semi-circle on AB as diameter. [Theorem 41.]

COROLLARY. *To cut off from a given circle a segment containing a given angle, it is enough to draw a tangent to the circle, and from the point of contact to draw a chord making with the tangent an angle equal to the given angle.*

It was proved on page 161 that

The locus of the vertices of triangles which stand on the same base and have a given vertical angle, is the arc of the segment standing on this base, and containing an angle equal to the given angle.

The following Problems are derived from this result by the Method of Intersection of Loci [page 93].

EXERCISES.

1. *Describe a triangle on a given base having a given vertical angle and having its vertex on a given straight line.*

2. *Construct a triangle having given the base, the vertical angle, and*
   (i) *one other side.*
   (ii) *the altitude.*
   (iii) *the length of the median which bisects the base.*
   (iv) *the foot of the perpendicular from the vertex to the base.*

3. *Construct a triangle having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle.*

[Let AB be the base, X the given point in it, and K the given angle. On AB describe a segment of a circle containing an angle equal to K; complete the $\odot^{ce}$ by drawing the arc APB. Bisect the arc APB at P: join PX, and produce it to meet the $\odot^{ce}$ at C. Then ABC is the required triangle.]

4. *Construct a triangle having given the base, the vertical angle, and the sum of the remaining sides.*

[Let AB be the given base, K the given angle, and H a line equal to the sum of the sides. On AB describe a segment containing an angle equal to K, also another segment containing an angle equal to half the $\angle$K. With centre A, and radius H, describe a circle cutting the arc of the latter segment at X and Y. Join AX (or AY) cutting the arc of the first segment at C. Then ABC is the required triangle.]

5. *Construct a triangle having given the base, the vertical angle, and the difference of the remaining sides.*

CIRCLES IN RELATION TO RECTILINEAL FIGURES.

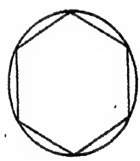
DEFINITIONS.

1. A **Polygon** is a rectilineal figure bounded by more than four sides.

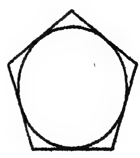
A Polygon of	*five* sides	is called a	**Pentagon,**
„	*six* sides	„	**Hexagon,**
„	*seven* sides	„	**Heptagon,**
„	*eight* sides	„	**Octagon,**
„	*ten* sides	„	**Decagon,**
„	*twelve* sides	„	**Dodecagon,**
„	*fifteen* sides	„	**Quindecagon.**

2. A Polygon is **Regular** when all its sides are equal, and all its angles are equal.

3. A rectilineal figure is said to be **inscribed** in a circle, when all its angular points are on the circumference of the circle; and a circle is said to be **circumscribed about** a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure.

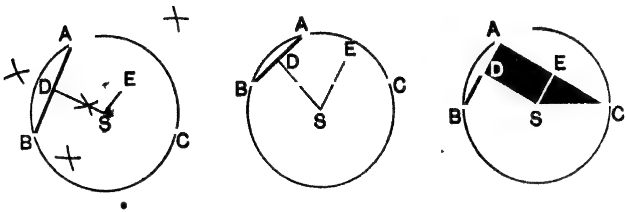


4. A circle is said to be **inscribed in** a rectilineal figure, when the circumference of the circle is touched by each side of the figure; and a rectilineal figure is said to be **circumscribed about** a circle, when each side of the figure is a tangent to the circle.



PROBLEM 25.

To circumscribe a circle about a given triangle.



Let ABC be the triangle, about which a circle is to be drawn.

Construction. Bisect AB and AC at rt. angles by DS and ES, meeting at S. *Prob.* 2.

Then S is the centre of the required circle.

Proof. Now every point in DS is equidistant from A and B; *Prob.* 14.

and every point in ES is equidistant from A and C;

$\therefore$ S is equidistant from A, B, and C.

With centre S, and radius SA describe a circle; this will pass through B and C, and is, therefore, the required circum-circle.

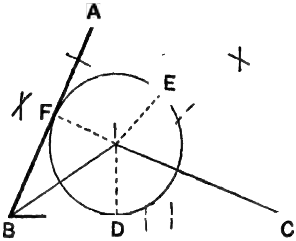
Obs. It will be found that if the given triangle is acute-angled, the centre of the circum-circle falls within it: if it is a right-angled triangle, the centre falls on the hypotenuse: if it is an obtuse-angled triangle, the centre falls without the triangle.

NOTE. From page 94 it is seen that if S is joined to the middle point of BC, then the joining line is perpendicular to BC.

Hence the perpendiculars drawn to the sides of a triangle from their middle points are concurrent, the point of intersection being the centre of the circle circumscribed about the triangle.

PROBLEM 26.

To inscribe a circle in a given triangle.



Let ABC be the triangle, in which a circle is to be inscribed.

Construction. Bisect the ∠ˢ ABC, ACB by the st. lines BI, CI, which intersect at I. *Prob.* 1.

Then I is the centre of the required circle.

Proof. From I draw ID, IE, IF perp. to BC, CA, AB.

Then every point in BI is equidistant from BC, BA; *Prob.* 15.

∴ ID = IF.

And every point in CI is equidistant from CB, CA;

∴ ID = IE.

∴ ID, IE, IF are all equal.

With centre I and radius ID draw a circle;
this will pass through the points E and F.

Also the circle will touch the sides BC, CA, AB,
because the angles at D, E, F are right angles.

∴ the ⊙ DEF is inscribed in the △ ABC.

NOTE. From II., p. 96 it is seen that if AI is joined, then AI bisects the angle BAC: hence it follows that

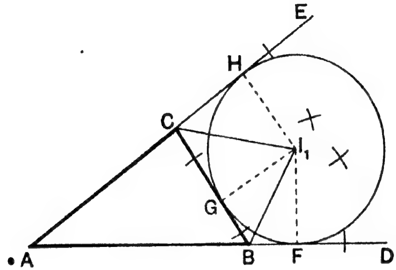
The bisectors of the angles of a triangle are concurrent, the point of intersection being the centre of the inscribed circle.

DEFINITION.

A circle which touches one side of a triangle and the other two sides produced is called an **escribed** circle of the triangle.

PROBLEM 27.

To draw an escribed circle of a given triangle.



Let ABC be the given triangle of which the sides AB, AC are produced to D and E.

It is required to describe a circle touching BC, *and* AB, AC *produced.*

Construction. Bisect the $\angle^s$ CBD, BCE by the st. lines BI_1, CI_1 which intersect at I_1.

Then I_1 is the centre of the required circle.

Proof. From I_1 draw I_1F, I_1G, I_1H perp. to AD, BC, AE.

Then every point in BI_1 is equidistant from BD, BC; *Prob.* 15.

$\therefore$ $I_1F = I_1G$.

Similarly $I_1G = I_1H$.

$\therefore$ I_1F, I_1G, I_1H are all equal.

With centre I_1 and radius I_1F describe a circle; this will pass through the points G and H.

Also the circle will touch AD, BC, and AE, because the angles at F, G, H are rt. angles.

$\therefore$ the $\odot$FGH is an escribed circle of the $\triangle$ABC.

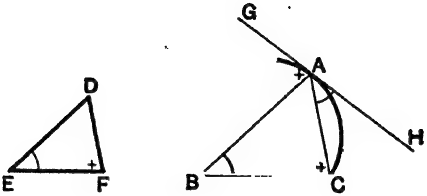
NOTE 1. It is clear that every triangle has three escribed circles. Their centres are known as the **Ex-centres**.

NOTE 2. It may be shewn, as in II., page 96, that if AI_1 is joined, then AI_1 bisects the angle BAC: hence it follows that

The bisectors of two exterior angles of a triangle and the bisector of the third angle are concurrent, the point of intersection being the centre of an escribed circle.

PROBLEM 28.

In a given circle to inscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle.

Analysis. A △ABC, equiangular to the △DEF, is inscribed in the circle, if from any point A on the $\bigcirc^{ce}$ two chords AB, AC can be so placed that, on joining BC, the ∠B = the ∠E, and the ∠C = the ∠F; for then the ∠A = the ∠D. *Theor.* 16.

Now the ∠B, in the segment ABC, suggests the *equal* angle between the chord AC and the tangent at its extremity (*Theor.* 49.); so that, if at A we draw the tangent GAH,

then the ∠HAC = the ∠E;

and similarly, the ∠GAB = the ∠F.

Reversing these steps, we have the following construction.

Construction. At any point A on the $\bigcirc^{ce}$ of the ⊙ABC draw the tangent GAH. *Prob.* 22.

At A make the ∠GAB equal to the ∠F,

and make the ∠HAC equal to the ∠E.

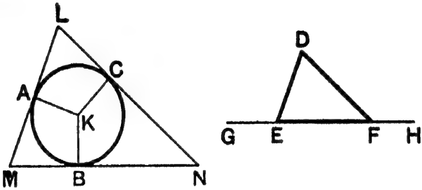
Join BC.

Then ABC is the required triangle.

NOTE. In drawing the figure on a larger scale the student should shew the construction lines for the tangent GAH and for the angles GAB, HAC. A similar remark applies to the next Problem.

PROBLEM 29.

About a given circle to circumscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle.

Analysis. Suppose LMN to be a circumscribed triangle in which the $\angle$ M = the $\angle$ E, the $\angle$ N = the $\angle$ F, and consequently, the $\angle$ L = the $\angle$ D.

Let us consider the radii KA, KB, KC, drawn to the points of contact of the sides; for the tangents LM, MN, NL could be drawn if we knew the relative positions of KA, KB, KC, that is, if we knew the $\angle^s$ BKA, BKC.

Now from the quad[l] BKAM, since the $\angle^s$ B and A are rt. $\angle^s$,

$$\text{the } \angle \text{BKA} = 180° - \text{M} = 180° - \text{E};$$

similarly $\quad$ the $\angle$ BKC $= 180° - \text{N} = 180° - \text{F}$.

Hence we have the following construction.

Construction. Produce EF both ways to G and H.

Find K the centre of the ⊙ ABC,
and draw any radius KB.
At K make the $\angle$ BKA equal to the $\angle$ DEG;
and make the $\angle$ BKC equal to the $\angle$ DFH.

Through A, B, C draw LM, MN, NL perp. to KA, KB, KC.
Then LMN is the required triangle.

[The student should now arrange the proof synthetically.]

EXERCISES.

On Circles and Triangles.

(Inscriptions and Circumscriptions.)

1. In a circle of radius 5 cm. inscribe an equilateral triangle; and about the same circle circumscribe a second equilateral triangle. In each case state and justify your construction.

2. Draw an equilateral triangle on a side of 8 cm., and find by calculation and measurement (to the nearest millimetre) the radii of the inscribed, circumscribed, and escribed circles.

Explain why the second and third radii are respectively double and treble of the first.

3. Draw triangles from the following data:

$$\text{(i)}\ a=2{\cdot}5'',\quad B=66^\circ,\quad C=50^\circ;$$
$$\text{(ii)}\ a=2{\cdot}5'',\quad B=72^\circ,\quad C=44^\circ;$$
$$\text{(iii)}\ a=2{\cdot}5'',\quad B=41^\circ,\quad C=23^\circ.$$

Circumscribe a circle about each triangle, and measure the radii to the nearest hundredth of an inch. Account for the three results being the same, by comparing the vertical angles.

4. In a circle of radius 4 cm. inscribe an equilateral triangle. Calculate the length of its side to the nearest millimetre; and verify by measurement.

Find the area of the inscribed equilateral triangle, and shew that it is one quarter of the circumscribed equilateral triangle.

5. In the triangle ABC, if I is the centre, and r the length of the radius of the in-circle, shew that

$$\triangle IBC=\tfrac{1}{2}ar;\quad \triangle ICA=\tfrac{1}{2}br;\quad \triangle IAB=\tfrac{1}{2}cr.$$

Hence prove that $\triangle ABC=\tfrac{1}{2}(a+b+c)r.$

Verify this formula by measurements for a triangle whose sides are 9 cm., 8 cm., and 7 cm.

6. If r_1 is the radius of the ex-circle opposite to A, prove that

$$\triangle ABC=\tfrac{1}{2}(b+c-a)r_1.$$

If $a=5$ cm., $b=4$ cm., $c=3$ cm., verify this result by measurement.

7. Find by measurement the circum-radius of the triangle ABC in which $a=6{\cdot}3$ cm., $b=3{\cdot}0$ cm., and $c=5{\cdot}1$ cm.

Draw and measure the perpendiculars from A, B, C to the opposite sides. If their lengths are represented by p_1, p_2, p_3, verify the following statement:

$$\text{circum-radius}=\frac{bc}{2p_1}=\frac{ca}{2p_2}=\frac{ab}{2p_3}.$$

EXERCISES.

On Circles and Squares.

(Inscriptions and Circumscriptions.)

1. Draw a circle of radius 1·5″, and find a construction for inscribing a square in it.

Calculate the length of the side to the nearest hundredth of an inch, and verify by measurement.

Find the area of the inscribed square.

2. Circumscribe a square about a circle of radius 1·5″, shewing all lines of construction.

Prove that the area of the square circumscribed about a circle is double that of the inscribed square.

3. Draw a square on a side of 7·5 cm., and state a construction for inscribing a circle in it.

Justify your construction by considerations of symmetry.

4. Circumscribe a circle about a square whose side is 6 cm.

Measure the diameter to the nearest millimetre, and test your drawing by calculation.

5. In a circle of radius 1·8″ inscribe a rectangle of which one side measures 3·0″. Find the approximate length of the other side.

Of all rectangles inscribed in the circle shew that the square has the greatest area.

6. A square and an equilateral triangle are inscribed in a circle. If a and b denote the lengths of their sides, shew that

$$3a^2 = 2b^2.$$

7. ABCD is a square inscribed in a circle, and P is any point on the arc AD: shew that the side AD subtends at P an angle three times as great as that subtended at P by any one of the other sides.

(Problems. State your construction, and give a theoretical proof.)

8. Circumscribe a rhombus about a given circle.

9. Inscribe a square in a given square ABCD, so that one of its angular points shall be at a given point X in AB.

10. In a given square inscribe the square of minimum area.

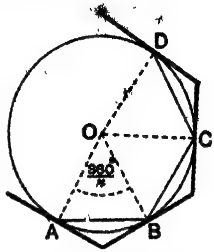
11. Describe (i) a circle, (ii) a square about a given rectangle.

12. Inscribe (i) a circle, (ii) a square in a given quadrant.

ON CIRCLES AND REGULAR POLYGONS.

PROBLEM 30.

To draw a regular polygon (i) **in** (ii) **about** *a given circle.*



Let AB, BC, CD, ... be consecutive sides of a regular polygon inscribed in a circle whose centre is O.

Then AOB, BOC, COD, ... are congruent isosceles triangles. And if the polygon has n sides, each of the $\angle^s$AOB, BOC, COD, ... $= \frac{360°}{n}$.

(i) Thus to **inscribe** a polygon of n sides in a given circle, draw an angle AOB at the centre equal to $\frac{360°}{n}$. This gives the length of a side AB; and chords equal to AB may now be set off round the circumference. The resulting figure will clearly be equilateral and equiangular.

(ii) To **circumscribe** a polygon of n sides about the circle, the points A, B, C, D, ... must be determined as before, and tangents drawn to the circle at these points. The resulting figure may readily be proved equilateral and equiangular.

NOTE. This method gives a *strict geometrical construction* only when the angle $\frac{360°}{n}$ can be drawn with ruler and compasses.

EXERCISES.

1. Give strict constructions for inscribing in a circle (radius 4 cm.) (i) a regular hexagon; (ii) a regular octagon; (iii) a regular dodecagon.

2. About a circle of radius 1·5″ circumscribe

(i) a regular hexagon; (ii) a regular octagon.

Test the constructions by measurement, and justify them by proof.

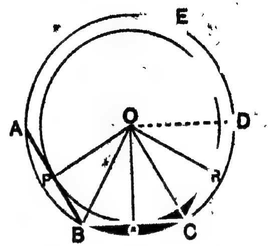
3. An equilateral triangle and a regular hexagon are inscribed in a given circle, and a and b denote the lengths of their sides: prove that

(i) area of triangle $= \frac{1}{2}$ (area of hexagon); (ii) $a^2 = 3b^2$.

4. By means of your protractor inscribe a regular heptagon in a circle of radius 2″. Calculate and measure one of its angles; and measure the length of a side.

PROBLEM 31.

To draw a circle (i) **in** (ii) **about** *a regular polygon.*



Let AB, BC, CD, DE, ... be consecutive sides of a regular polygon of n sides.

Bisect the $\angle$ˢ ABC, BCD by BO, CO meeting at O.

Then O is the centre both of the inscribed and circumscribed circle.

Outline of Proof. Join OD; and from the congruent $\triangle$ˢ OCB, OCD, shew that OD bisects the $\angle$CDE. Hence we conclude that

All the bisectors of the angles of the polygon meet at O.

(i) Prove that OB = OC = OD = ...; from Theorem 6.
Hence O is the circum-centre.

(ii) Draw OP, OQ, OR, ... perp. to AB, BC, CD, ...
Prove that OP = OQ = OR = ...; from the congruent $\triangle$ˢ OBP, OBQ, ...
Hence O is the in-centre.

EXERCISES.

1. Draw a regular hexagon on a side of 2·0". Draw the inscribed and circumscribed circles. Calculate and measure their diameters to the nearest hundredth of an inch.

2. Shew that the area of a regular hexagon inscribed in a circle is three-fourths of that of the circumscribed hexagon.

Find the area of a hexagon inscribed in a circle of radius 10 cm. to the nearest tenth of a sq. cm.

3. If ABC is an isosceles triangle inscribed in a circle, having each of the angles B and C double of the angle A; shew that BC is a side of a regular pentagon inscribed in the circle.

4. On a side of 4 cm. construct (without protractor)

(i) a regular hexagon; (ii) a regular octagon.

In each case find the approximate area of the figure.

THE CIRCUMFERENCE OF A CIRCLE.

By experiment and measurement it is found that the length of the circumference of a circle is roughly $3\frac{1}{7}$ times the length of its diameter: that is to say

$$\frac{circumference}{diameter} = 3\tfrac{1}{7} \text{ nearly};$$

and it can be proved that this is the same for all circles.

A more correct value of this ratio is found by theory to be 3·1416; while correct to 7 places of decimals it is 3·1415926. Thus the value $3\frac{1}{7}$ (or 3·14*28*) is too great, and correct to 2 places only.

The ratio which the circumference of any circle bears to its diameter is denoted by the Greek letter π; so that

$$circumference = diameter \times \pi.$$

Or, if r denotes the radius of the circle,

$$circumference = 2r \times \pi = 2\pi r;$$

where to π we are to give one of the values $3\frac{1}{7}$, 3·1416, or 3·1415926, according to the degree of accuracy required in the final result.

NOTE. The theoretical methods by which π is evaluated to any required degree of accuracy cannot be explained at this stage, but its value may be easily verified by experiment to two decimal places.

For example: round a cylinder wrap a strip of paper so that the ends overlap. At any point in the overlapping area prick a pin through both folds. Unwrap and straighten the strip, then measure the distance between the pin holes: this gives the length of the circumference. Measure the diameter, and divide the first result by the second.

Ex. 1. From these data find and record the value of π.

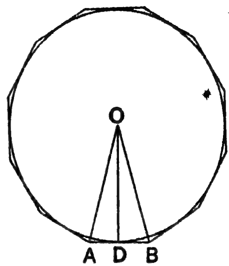
Find the mean of the three results.

CIRCUMFERENCE.	DIAMETER.	VALUE OF π.
16·0 cm.	5·1 cm.	
8·8″	2·8″	
13·5″	4·3″	

Ex. 2. A fine thread is wound evenly round a cylinder, and it is found that the length required for 20 complete turns is 75·4″. The diameter of the cylinder is 1·2″: find roughly the value of π.

Ex. 3. A bicycle wheel, 28″ in diameter, makes 400 revolutions in travelling over 977 yards. From this result estimate the value of π.

THE AREA OF A CIRCLE.



Let AB be a side of a polygon of n sides circumscribed about a circle whose centre is O and radius r. Then we have

Area of polygon

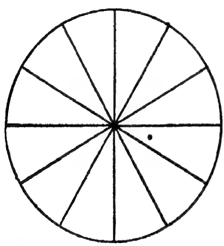
$$= n \,.\, \triangle \mathrm{AOB}$$
$$= n \,.\, \tfrac{1}{2}\mathrm{AB} \times \mathrm{OD}$$
$$= \tfrac{1}{2} \,.\, n\mathrm{AB} \times r$$
$$= \tfrac{1}{2}(\textit{perimeter of polygon}) \times r\,;$$

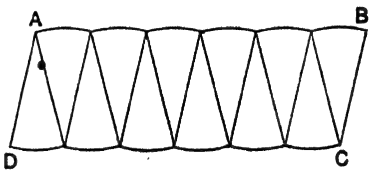
and this is true however many sides the polygon may have.

Now if the number of sides is increased without limit, the perimeter and area of the polygon may be made to differ from the circumference and area of the circle by quantities smaller than any that can be named; hence ultimately

$$\textit{Area of circle} = \tfrac{1}{2} \,.\, \textit{circumference} \times r$$
$$= \tfrac{1}{2} \,.\, 2\pi r \times r$$
$$= \pi r^2.$$

ALTERNATIVE METHOD.





Suppose the circle divided into any even number of sectors having equal central angles: denote the number of sectors by n.

Let the sectors be placed side by side as represented in the diagram;

then the area of the circle = the area of the fig. ABCD;

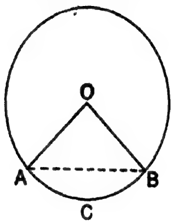
and this is true however great n may be.

Now as the number of sectors is increased, each arc is decreased; so that (i) the outlines AB, CD tend to become *straight*, and

Thus when n is increased without limit, the fig. ABCD ultimately becomes a *rectangle*, whose length is the *semi-circumference of the circle*, and whose breadth is its *radius*.

$$\therefore \textit{Area of circle} = \tfrac{1}{2}\,.\,\textit{circumference} \times \textit{radius}$$
$$= \tfrac{1}{2}\,.\,2\pi r \times r = \pi r^2.$$

THE AREA OF A SECTOR.



If two radii of a circle make an angle of 1°, they cut off

(i) an arc whose length $= \frac{1}{360}$ of the circumference;

and (ii) a sector whose area $= \frac{1}{360}$ of the circle;

∴ if the angle AOB contains D degrees, then

(i) *the arc* AB $= \dfrac{D}{360}$ *of the circumference*;

(ii) *the sector* AOB $= \dfrac{D}{360}$ *of the area of the circle*

$= \dfrac{D}{360}$ *of* ($\frac{1}{2}$ *circumference* × *radius*)

$= \frac{1}{2}$. *arc* AB × *radius*.

THE AREA OF A SEGMENT.

The area of a minor segment is found by subtracting from the corresponding sector the area of the triangle formed by the chord and the radii. Thus

Area of segment ABC = *sector* OACB − *triangle* AOB.

The area of a major segment is most simply found by subtracting the area of the corresponding minor segment from the area of the circle.

EXERCISES.

[*In each case choose the value of* π *so as to give a result of the assignea degree of accuracy.*]

1. Find to the nearest millimetre the circumferences of the circles whose radii are (i) 4·5 cm. (ii) 100 cm.

2. Find to the nearest hundredth of a square inch the areas of the circles whose radii are (i) 2·3″. (ii) 10·6″.

3. Find to two places of decimals the circumference and area of a circle inscribed in a square whose side is 3·6 cm.

4. In a circle of radius 7·0 cm. a square is described: find to the nearest square centimetre the difference between the areas of the circle and the square.

5. Find to the nearest hundredth of a square inch the area of the circular ring formed by two concentric circles whose radii are 5·7″ and 4·3″.

6. Shew that the area of a ring lying between the circumferences of two concentric circles is equal to the area of a circle whose radius is the length of a tangent to the inner circle from any point on the outer.

7. A rectangle whose sides are 8·0 cm. and 6·0 cm. is inscribed in a circle. Calculate to the nearest tenth of a square centimetre the total area of the four segments outside the rectangle.

8. Find to the nearest tenth of an inch the side of a square whose area is equal to that of a circle of radius 5″.

9. A circular ring is formed by the circumference of two concentric circles. The area of the ring is 22 square inches, and its width is 1·0″; taking π as $\frac{22}{7}$, find approximately the radii of the two circles.

10. Find to the nearest hundredth of a square inch the difference between the areas of the circumscribed and inscribed circles of an equilateral triangle each of whose sides is 4″.

11. Draw on squared paper two circles whose centres are at the points (1·5″, 0) and (0, ·8″), and whose radii are respectively ·7″ and 1·0″. Prove that the circles touch one another, and find approximately their circumferences and areas.

12. Draw a circle of radius 1·0″ having the point (1·6″, 1·2″) as centre. Also draw two circles with the origin as centre and of radii 1·0″ and 3·0″ respectively. Shew that each of the last two circles touches the first.

EXERCISES.

ON THE INSCRIBED, CIRCUMSCRIBED, AND ESCRIBED CIRCLES OF A TRIANGLE.

(*Theoretical.*)

1. Describe a circle to touch two parallel straight lines and a third straight line which meets them. Shew that two such circles can be drawn, and that they are equal.

2. *Triangles which have equal bases and equal vertical angles have equal circumscribed circles.*

3. ABC is a triangle, and I, S are the centres of the inscribed and circumscribed circles; if A, I, S are collinear, shew that AB=AC.

4. The sum of the diameters of the inscribed and circumscribed circles of a right-angled triangle is equal to the sum of the sides containing the right angle.

5. If the circle inscribed in the triangle ABC touches the sides at D, E, F; shew that the angles of the triangle DEF are respectively

$$90-\frac{A}{2},\ 90-\frac{B}{2},\ 90-\frac{C}{2}.$$

6. It I is the centre of the circle inscribed in the triangle ABC, and I_1 the centre of the escribed circle which touches BC; shew that I, B, I_1, C are concyclic.

7. In any triangle the difference of two sides is equal to the difference of the segments into which the third side is divided at the point of contact of the inscribed circle.

8. In the triangle ABC, I and S are the centres of the inscribed and circumscribed circles: shew that IS subtends at A an angle equal to half the difference of the angles at the base of the triangle.

Hence shew that if AD is drawn perpendicular to BC, then AI is the bisector of the angle DAS.

9. The diagonals of a quadrilateral ABCD intersect at O: shew that the centres of the circles circumscribed about the four triangles AOB, BOC, COD, DOA are at the angular points of a parallelogram.

10. In any triangle ABC, if I is the centre of the inscribed circle, and if AI is produced to meet the circumscribed circle at O; shew that O is the centre of the circle circumscribed about the triangle BIC.

11. Given the base, altitude, and the radius of the circumscribed circle; construct the triangle.

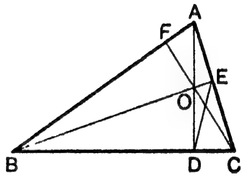
12. Three circles whose centres are A, B, C touch one another externally two by two at D, E, F: shew that the inscribed circle of the triangle ABC is the circumscribed circle of the triangle DEF.

THEOREMS AND EXAMPLES ON CIRCLES AND TRIANGLES.

THE ORTHOCENTRE OF A TRIANGLE.

I. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*

In the $\triangle$ABC, let AD, BE be the perp^s drawn from A and B to the opposite sides; and let them intersect at O.



Join CO; and produce it to meet AB at F.

It is required to shew that CF *is perp. to* AB.

Join DE.

Then, because the $\angle^s$ OEC, ODC are rt. angles,

$\therefore$ the points O, E, C, D are concyclic:

$\therefore$ the $\angle$DEC = the $\angle$DOC, in the same segment;

= the vert. opp. $\angle$FOA.

Again, because the $\angle^s$ AEB, ADB are rt. angles,

$\therefore$ the points A, E, D, B are concyclic:

$\therefore$ the $\angle$DEB = the $\angle$DAB, in the same segment.

$\therefore$ the sum of the $\angle^s$ FOA, FAO = the sum of the $\angle^s$ DEC, DEB

= a rt. angle:

$\therefore$ the remaining $\angle$AFO = a rt. angle: *Theor.* 16.

that is, CF is perp. to AB.

Hence the three perp AD, BE, CF meet at the point O.

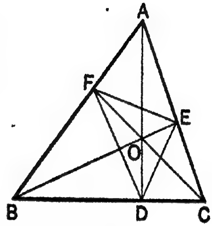
Q.E.D.

DEFINITIONS.

(i) The intersection of the perpendiculars drawn from the vertices of a triangle to the opposite sides is called its **orthocentre.**

(ii) The triangle formed by joining the feet of the perpendiculars is called the **pedal** or **orthocentric triangle.**

II. *In an acute-angled triangle the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle through which they pass.*



In the acute-angled △ ABC, let AD, BE, CF be the perp^s drawn from the vertices to the opposite sides, meeting at the orthocentre O; and let DEF be the pedal triangle.

It is required to prove that

AD, BE, CF *bisect respectively*

the ∠^{s} FDE, DEF, EFD.

It may be shewn, as in the last theorem, that the points O, D, C, E are concyclic;

∴ the ∠ODE = the ∠OCE, in the same segment.

Similarly the points O, D, B, F are concyclic;

∴ the ∠ODF = the ∠OBF, in the same segment.

But the ∠OCE = the ∠OBF, each being the comp^t of the ∠BAC.

∴ the ∠ODE = the ∠ODF.

Similarly it may be shewn that the ∠^{s} DEF, EFD are bisected by BE and CF. Q.E.D.

Corollary. (i) *Every two sides of the pedal triangle are equally inclined to that side of the original triangle in which they meet.*

For the ∠EDC = the comp^t of the ∠ODE
= the comp^t of the ∠OCE
= the ∠BAC.

Similarly it may be shewn that the ∠FDB = the ∠BAC,

∴ the ∠EDC = the ∠FDB = the ∠A.

In like manner it may be proved that

the ∠DEC = the ∠FEA = the ∠B,

and the ∠DFB = the ∠EFA = the ∠C.

Corollary. (ii) *The triangles* DEC, AEF, DBF *are equiangular to one another and to the triangle* ABC.

Note. If the angle BAC is *obtuse*, then the perpendiculars BE, CF bisect *externally* the corresponding angles of the pedal triangle.

EXERCISES.

1. *If* O *is the orthocentre of the triangle* ABC *and if the perpendicular* AD *is produced to meet the circum-circle in* G, *prove that* OD = DG.

2. *In an acute-angled triangle the three sides are the external bisectors of the angles of the pedal triangle: and in an obtuse-angled triangle the sides containing the obtuse angle are the internal bisectors of the corresponding angles of the pedal triangle.*

3. *If* O *is the orthocentre of the triangle* ABC, *shew that the angles* BOC, BAC *are supplementary.*

4. *If* O *is the orthocentre of the triangle* ABC, *then any one of the four points* O, A, B, C *is the orthocentre of the triangle whose vertices are the other three.*

5. *The three circles which pass through two vertices of a triangle and its orthocentre are each equal to the circum-circle of the triangle.*

6. D, E are taken on the circumference of a semi-circle described on a given straight line AB: the chords AD, BE and AE, BD intersect (produced if necessary) at F and G: shew that FG is perpendicular to AB.

7. ABC is a triangle, O is its orthocentre, and AK a diameter of the circum-circle: shew that BOCK is a parallelogram.

8. The orthocentre of a triangle is joined to the middle point of the base, and the joining line is produced to meet the circum-circle: prove that it will meet it at the same point as the diameter which passes through the vertex.

9. The perpendicular from the vertex of a triangle on the base, and the straight line joining the orthocentre to the middle point of the base, are produced to meet the circum-circle at P and Q: shew that PQ is parallel to the base.

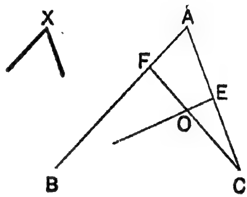
10. *The distance of each vertex of a triangle from the orthocentre is double of the perpendicular drawn from the centre of the circum-circle to the opposite side.*

11. Three circles are described each passing through the orthocentre of a triangle and two of its vertices: shew that the triangle formed by joining their centres is equal in all respects to the original triangle.

12. Construct a triangle, having given a vertex, the orthocentre, and the centre of the circum-circle.

LOCI.

III. *Given the base and vertical angle of a triangle, find the locus of its orthocentre.*



Let BC be the given base, and X the given angle; and let BAC be any triangle on the base BC, having its vertical ∠A equal to the ∠X.

Draw the perp^s BE, CF, intersecting at the orthocentre O.

It is required to find the locus of O.

Proof. Since the ∠^{s} OFA, OEA are rt. angles,

∴ the points O, F, A, E are concyclic;

∴ the ∠FOE is the supplement of the ∠A:

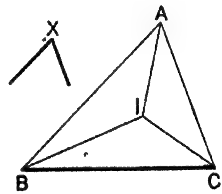
∴ the vert. opp. ∠BOC is the supplement of the ∠A.

But the ∠A is constant, being always equal to the ∠X;

∴ its supplement is constant;

that is, the △BOC has a fixed base, and constant vertical angle; hence the locus of its vertex O is the arc of a segment of which BC is the chord.

IV. *Given the base and vertical angle of a triangle, find the locus of the in-centre.*



Let BAC be any triangle on the given base BC, having its vertical angle equal to the given ∠X; and let AI, BI, CI be the bisectors of its angles. Then I is the in-centre.

It is required to find the locus of I.

Proof. Denote the angles of the △ABC by A, B, C; and let the ∠BIC be denoted by I.

Then from the △BIC,

(i) $I+\frac{1}{2}B+\frac{1}{2}C=$ two rt. angles; *Theor.* 16.

and from the △ABC,

$A+B+C=$ two rt. angles;

(ii) so that $\frac{1}{2}A+\frac{1}{2}B+\frac{1}{2}C=$ one rt. angle.

∴, taking the differences of the equals in (i) and (ii),

$I-\frac{1}{2}A=$ one rt. angle:

or, $I=$ one rt. angle $+\frac{1}{2}A$.

But A is constant, being always equal to the ∠X;

∴ I is constant:

∴ the locus of I is the arc of a segment on the fixed chord BC.

EXERCISES ON LOCI.

1. Given the base BC and the vertical angle A of a triangle; find the locus of the ex-centre opposite A.

2. Through the extremities of a given straight line AB any two parallel straight lines AP, BQ are drawn; find the locus of the intersection of the bisectors of the angles PAB, QBA.

3. Find the locus of the middle points of chords of a circle drawn through a fixed point.

Distinguish between the cases when the given point is within, on, or without the circumference.

4. Find the locus of the points of contact of tangents drawn from a fixed point to a system of concentric circles.

5. Find the locus of the intersection of straight lines which pass through two fixed points on a circle and intercept on its circumference an arc of constant length.

6. A and B are two fixed points on the circumference of a circle, and PQ is any diameter: find the locus of the intersection of PA and QB.

7. BAC is any triangle described on the fixed base BC and having a constant vertical angle; and BA is produced to P, so that BP is equal to the sum of the sides containing the vertical angle: find the locus of P.

8. AB is a fixed chord of a circle, and AC is a moveable chord passing through A: if the parallelogram CB is completed, find the locus of the intersection of its diagonals.

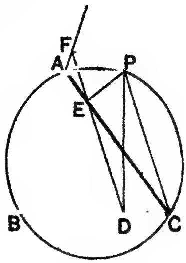
9. A straight rod PQ slides between two rulers placed at right angles to one another, and from its extremities PX, QX are drawn perpendicular to the rulers: find the locus of X.

10. Two circles intersect at A and B, and through P, any point on the circumference of one of them, two straight lines PA, PB are drawn, and produced if necessary, to cut the other circle at X and Y: find the locus of the intersection of AY and BX.

11. Two circles intersect at A and B; HAK is a fixed straight line drawn through A and terminated by the circumferences, and PAQ is any other straight line similarly drawn: find the locus of the intersection of HP and QK.

SIMSON'S LINE.

V. *The feet of the perpendiculars drawn to the three sides of a triangle from any point on its circum-circle are collinear.*



Let P be any point on the circum-circle of the △ ABC; and let PD, PE, PF be the perps. drawn from P to the sides.

It is required to prove that the points D, E, F *are collinear.*

Join FE and ED:

then FE and ED will be shewn to be in the same straight line.

Join PA, PC.

Proof. Because the ∠s PEA, PFA are rt. angles,

∴ the points P, E, A, F are concyclic:

∴ the ∠PEF = the ∠PAF, in the same segment
= the supp^t of the ∠PAB
= the ∠PCD,

since the points A, P, C, B are concyclic.

Again because the ∠s PEC, PDC are rt. angles,

∴ the points P, E, D, C are concyclic.

∴ the ∠PED = the supp^t of the ∠PCD
= the supp^t of the ∠PEF.

∴ FE and ED are in one st. line.

Obs. The line FED is known as the **Pedal** or **Simson's Line** of the triangle ABC for the point P.

EXERCISES.

1. From any point P on the circum-circle of the triangle ABC, perpendiculars PD, PF are drawn to BC and AB: if FD, or FD produced, cuts AC at E, shew that PE is perpendicular to AC.

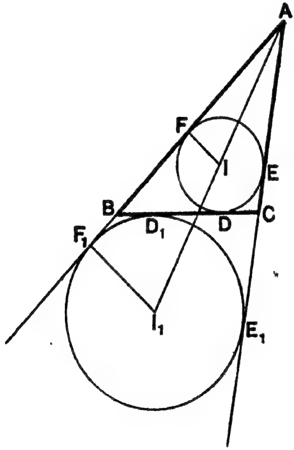
2. Find the locus of a point which moves so that if perpendiculars are drawn from it to the sides of a given triangle, their feet are collinear.

3. ABC and AB'C' are two triangles with a common angle, and their circum-circles meet again at P; shew that the feet of perpendiculars drawn from P to the lines AB, AC, BC, B'C' are collinear.

4. A triangle is inscribed in a circle, and any point P on the circumference is joined to the orthocentre of the triangle: shew that this joining line is bisected by the pedal of the point P.

THE TRIANGLE AND ITS CIRCLES.

VI. D, E, F *are the points of contact of the inscribed circle of the triangle* ABC, *and* D_1, E_1, F_1 *the points of contact of the escribed circle, which touches* BC *and the other sides produced:* a, b, c *denote the length of the sides* BC, CA, AB; s *the semi-perimeter of the triangle, and* r, r_1 *the radii of the inscribed and escribed circles.*



Prove the following equalities:

(i) $AE = AF = s - a$,
$BD = BF = s - b$,
$CD = CE = s - c$.

(ii) $AE_1 = AF_1 = s$.

(iii) $CD_1 = CE_1 = s - b$,
$BD_1 = BF_1 = s - c$.

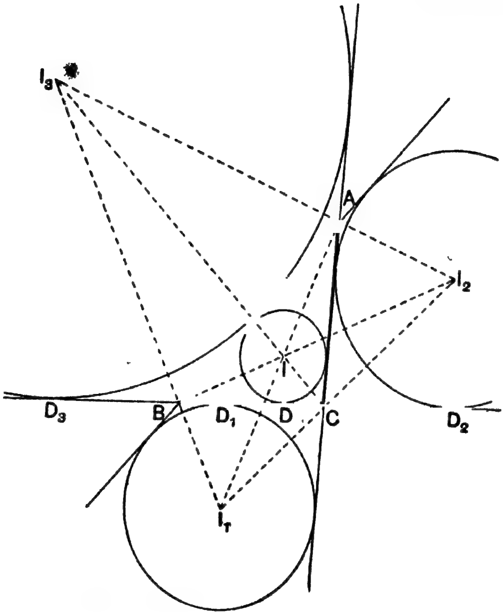
(iv) $CD = BD_1$, and $BD = CD_1$.

(v) $EE_1 = FF_1 = a$.

(vi) The area of the $\triangle ABC = rs$
$= r_1(s - a)$.

(vii) Draw the above figure in the case when C is a right angle, and prove that $r = s - c$; $r_1 = s - b$.

VII. *In the triangle* ABC, I *is the centre of the inscribed circle, and* I_1, I_2, I_3 *the centres of the escribed circles touching respectively the sides* BC, CA, AB *and the other sides produced.*



Prove the following properties:

(i) *The points* A, I, I_1 *are collinear: so are* B, I, I_2; *and* C, I, I_3.

(ii) *The points* I_2, A, I_3 *are collinear; so are* I_3, B, I_1; *and* I_1, C, I_2.

(iii) *The triangles* BI_1C, CI_2A, AI_3B *are equiangular to one another.*

(iv) *The triangle* $I_1I_2I_3$ *is equiangular to the triangle formed by joining the points of contact of the inscribed circle.*

(v) *Of the four points* I, I_1, I_2, I_3, *each is the orthocentre of the triangle whose vertices are the other three.*

(vi) *The four circles, each of which passes through three of the points* I, I_1, I_2, I_3 *are all equal.*

EXERCISES.

1. With the figure given on page 214, shew that if the circles whose centres are I, I_1, I_2, I_3 touch BC at D, D_1, D_2, D_3, then

(i) $DD_2 = D_1D_3 = b$. (ii) $DD_3 = D_1D_2 = c$.

(iii) $D_2D_3 = b + c$. (iv) $DD_1 = b \sim c$.

2. *Shew that the orthocentre and vertices of a triangle are the centres of the inscribed and escribed circles of the pedal triangle.*

3. *Given the base and vertical angle of a triangle, find the locus of the centre of the escribed circle which touches the base.*

4. *Given the base and vertical angle of a triangle, shew that the centre of the circum-circle is fixed.*

5. Given the base BC, and the vertical angle A of the triangle, find the locus of the centre of the escribed circle which touches AC.

6. Given the base, the vertical angle, and the point of contact with the base of the in-circle; construct the triangle.

7. Given the base, the vertical angle, and the point of contact with the base, or base produced, of an escribed circle; construct the triangle.

8. *I is the centre of the circle inscribed in a triangle, and I_1, I_2, I_3 the centres of the escribed circles; shew that II_1, II_2, II_3 are bisected by the circumference of the circum-circle.*

9. ABC is a triangle, and I_2, I_3 the centres of the escribed circles which touch AC, and AB respectively: shew that the points B, C, I_2, I_3 lie upon a circle whose centre is on the circumference of the circum-circle of the triangle ABC.

10. With three given points as centres describe three circles touching one another two by two. How many solutions will there be?

11. Given the centres of the three escribed circles; construct the triangle.

12. Given the centre of the inscribed circle, and the centres of two escribed circles; construct the triangle.

13. Given the vertical angle, perimeter, and radius of the inscribed circle; construct the triangle.

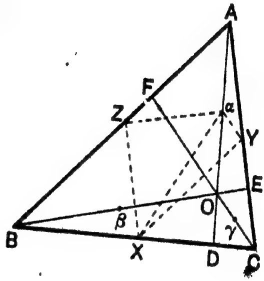
14. Given the vertical angle, the radius of the inscribed circle, and the length of the perpendicular from the vertex to the base; construct the triangle.

15. In a triangle ABC, I is the centre of the inscribed circle; shew that the centres of the circles circumscribed about the triangles BIC, CIA, AIB lie on the circumference of the circle circumscribed about the given triangle.

THE NINE-POINTS CIRCLE.

VIII. *In any triangle the middle points of the sides, the feet of the perpendiculars from the vertices to the opposite sides, and the middle points of the lines joining the orthocentre to the vertices are concyclic.*

In the △ABC, let X, Y, Z be the middle points of the sides BC, CA, AB; let D, E, F be the feet of the perp^s to these sides from A, B, C; let O be the orthocentre, and α, β, γ the middle points of OA, OB, OC.



It is required to prove that the nine points X, Y, Z, D, E, F, α, β, γ *are concyclic.*

Join XY, XZ, Xα, Yα, Zα.

Now from the △ABO,
since AZ=ZB, and Aα=αO,
∴ Zα is par^l to BO. Ex. 2, p. 64.

And from the △ABC, since BZ=ZA, and BX=XC,
∴ ZX is par^l to AC.
But BO produced makes a rt. angle with AC;
∴ the ∠XZα is a rt. angle.

Similarly, the ∠XYα is a rt. angle.
∴ the points X, Z, α, Y are concyclic:
that is, α lies on the O^{ce} of the circle which passes through X, Y, Z; and Xα is a diameter of this circle.

Similarly it may be shewn that β and γ lie on the O^{ce} of this circle.

Again, since αDX is a rt. angle,
∴ the circle on Xα as diameter passes through D.

Similarly it may be shewn that E and F lie on the O^{ce} of this circle;
∴ the points X, Y, Z, D, E, F, α, β, γ are concyclic. Q.E.D.

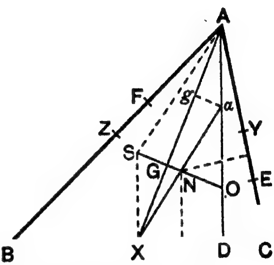
Obs. From this property the circle which passes through the middle points of the sides of a triangle is called the **Nine-Points Circle**; many of its properties may be derived from the fact of its being the circum-circle of the pedal triangle.

To prove that

(i) *the centre of the nine-points circle is the middle point of the straight line which joins the orthocentre to the circum-centre.*

(ii) *the radius of the nine-points circle is half the radius of the circum-circle.*

(iii) *the centroid is collinear with the circum-centre, the nine-points centre, and the orthocentre.*



In the $\triangle$ABC, let X, Y, Z be the middle points of the sides; D, E, F the feet of the perps; O the orthocentre; S and N the centres of the circumscribed and nine-points circles respectively.

(i) *To prove that* N *is the middle point of* SO.

It may be shewn that the perp. to XD from its middle point bisects SO; *Theor.* 22.

Similarly the perp. to EY at its middle point bisects SO:

that is, these perps intersect at the middle point of SO:

And since XD and EY are chords of the nine-points circle,

$\therefore$ the intersection of the lines which bisect XD and EY at rt. angles is its centre: *Theor.* 31, *Cor.* 1.

$\therefore$ the centre N is the middle point of SO. Q.E.D.

(ii) *To prove that the radius of the nine-points circle is half the radius of the circum-circle.*

By the last Proposition, Xa is a diameter of the nine-points circle.

$\therefore$ the middle point of Xa is its centre:

but the middle point of SO is also the centre of the nine-points circle. (*Proved.*)

Hence Xa and SO bisect one another at N.

Then from the $\triangle$s SNX, ONa,

because $\begin{cases} \text{SN} = \text{ON}, \\ \text{and NX} = \text{N}a, \\ \text{and the } \angle \text{SNX} = \text{the } \angle \text{ON}a; \end{cases}$

$\therefore$ SX = Oa
= Aa.

And SX is also parl to Aa,

$\therefore$ SA = Xa.

But SA is a *radius* of the circum-circle;

and Xa is a *diameter* of the nine-points circle;

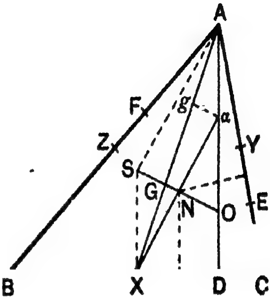
the radius of the nine-points circle is half the radius of the circum-circle. [See also p. 267, Examples 2 and 3.] Q.E.D.

(iii) *To prove that the centroid is collinear with points* S, N, O.

Join AX and draw ag par[l] to SO.
Let AX meet SO at G.

Then from the $\triangle$ AGO, since $A\alpha = \alpha O$,
and ag is par[l] to OG,
$\therefore$ $Ag = gG$. *Ex.* 1, p. 64.

And from the $\triangle X\alpha g$, since $\alpha N = NX$,
and NG is par[l] to ag,
$\therefore$ $gG = GX$.
$\therefore$ $AG = \frac{2}{3}$ of AX;
$\therefore$ G is the centroid of the triangle ABC.
Theor. III., *Cor.*, p. 97.
That is, the centroid is collinear with the points S, N, O. Q.E.D.



EXERCISES.

1. *Given the base and vertical angle of a triangle, find the locus of the centre of the nine-points circle.*

2. The nine-points circle of any triangle ABC, whose orthocentre is O, is also the nine-points circle of each of the triangles AOB, BOC, COA.

3. If I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles of a triangle ABC, then the circle circumscribed about ABC is the nine-points circle of each of the four triangles formed by joining three of the points I, I_1, I_2, I_3.

4. All triangles which have the same orthocentre and the same circumscribed circle, have also the same nine-points circle.

5. Given the base and vertical angle of a triangle, shew that one angle and one side of the pedal triangle are constant.

6. Given the base and vertical angle of a triangle, find the locus of the centre of the circle which passes through the three escribed centres.

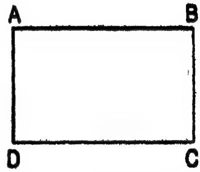
NOTE. For some other important properties of the Nine-points Circle see Ex. 54, page 310.

PART IV.

ON SQUARES AND RECTANGLES IN CONNECTION WITH THE SEGMENTS OF A STRAIGHT LINE.

THE GEOMETRICAL EQUIVALENTS OF CERTAIN ALGEBRAICAL FORMULÆ.

DEFINITIONS.



1. A rectangle ABCD is said to be **contained** by two adjacent sides AB, AD; for these sides fix its size and shape.

A rectangle whose adjacent sides are AB, AD is denoted by *the rect.* AB, AD; this is equivalent to the product AB . AD.

Similarly a square drawn on the side AB is denoted by *the sq. on* AB, or AB^2.

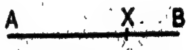


Fig. 1.

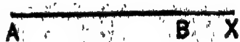


Fig. 2.

2. If a point X is taken in a straight line AB, or in AB produced, then X is said to divide AB into the two segments AX, XB; the segments being in either case *the distances of the dividing point* X *from the extremities of the given line* AB.

In Fig. 1, AB is said to be divided **internally** at X.

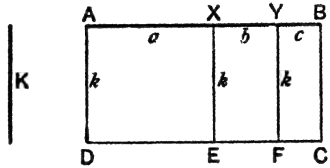
In Fig. 2, AB .... divided **externally** at X.

Obs. In internal division the given line AB is the *sum* of the segments AX, XB.

In external division the given line AB is the *difference* of the segments AX, XB.

Theorem 50. [Euclid II. 1.]

If of two straight lines, one is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.



Let AB and K be the two given st. lines, and let AB be divided into any number of parts AX, XY, YB, which contain respectively a, b, and c units of length; so that AB contains $a+b+c$ units.

Let the line K contain k units of length.

It is required to prove that

the rect. AB, K = rect. AX, K + rect. XY, K + rect. YB, K,

namely that

$$(a+b+c)k = \quad ak \quad + \quad bk \quad + \quad ck.$$

Construction. Draw AD perp. to AB and equal to K.

Through D draw DC par[l] to AB.

Through X, Y, B draw XE, YF, BC par[l] to AD.

Proof. The fig. AC = the fig. AE + the fig. XF + the fig. YC; and of these, by construction,

fig. AC = rect. AB, K; and contains $(a+b+c)k$ units of area:

$\begin{cases} \text{fig. AE = rect. AX, K; and contains } ak \text{ units of area;} \\ \text{fig. XF = rect. XY, K; } \ldots\ldots\ldots\ldots \; bk \; \ldots\ldots\ldots\ldots, \\ \text{fig. YC = rect. YB, K; } \ldots\ldots\ldots\ldots \; ck \; \ldots\ldots\ldots\ldots. \end{cases}$

Hence

the rect. AB, K = rect. AX, K + rect. XY, K + rect. YB, K;

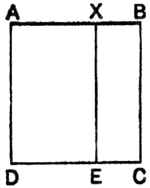
or, $$(a+b+c)k = \quad ak \quad + \quad bk \quad + \quad ck.$$

Q.E.D.

* COROLLARIES. [Euclid II. 2 and 3.]

Two special cases of this Theorem deserve attention.

(i) When AB is divided only at one point X, and when the undivided line AD is equal to AB.



Then the sq. on AB = the rect. AB, AX + the rect. AB, XB.

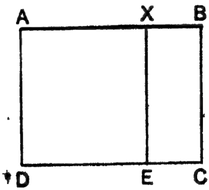
That is,

The square on the given line is equal to the sum of the rectangles contained by the whole line and each of the segments.

Or thus:

$$\begin{aligned} AB^2 &= AB \,.\, AB \\ &= AB(AX + XB) \\ &= AB \,.\, AX + AB \,.\, XB. \end{aligned}$$

(ii) When AB is divided at one point X, and when the undivided line AD is equal to one segment AX.



Then the rect. AB, AX = the sq. on AX + the rect. AX, XB.

That is,

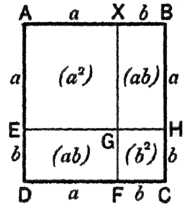
The rectangle contained by the whole line and one segment is equal to the square on that segment with the rectangle contained by the two segments.

Or thus:

$$\begin{aligned} AB \,.\, AX &= (AX + XB)AX \\ &= AX^2 + AX \,.\, XB. \end{aligned}$$

THEOREM 51. [Euclid II. 4.]

If a straight line is divided **internally** *at any point, the square on the given line is equal to the sum of the squares on the two segments* **together with** *twice the rectangle contained by the segments.*



Let AB be the given st. line divided *internally* at X; and let the segments AX, XB contain a and b units of length respectively.

Then AB is the *sum* of the segments AX, XB, and therefore contains $a+b$ units.

It is required to prove that

$$AB^2 = AX^2 + XB^2 + 2AX \cdot XB;$$

namely that

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

Construction. On AB describe a square ABCD. From AD cut off AE equal to AX, or a. Then $ED = XB = b$. Through E and X draw EH, XF par' respectively to AB, AD and meeting at G.

Proof. Then the fig. AC = the figs. AG, GC + the figs. EF, XH.

And of these, by construction,

fig. AC is the sq. on AB, and contains $(a+b)^2$ units of area;

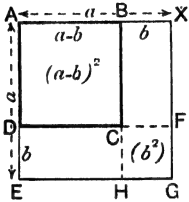
fig. AG = sq. on AX, and contains a^2 units of area;
fig. GC = sq. on XB, ... b^2 ....;
fig. EF = rect. EG, ED
  = rect. AX, XB .... ab ....;
fig. XH = rect. GX, XB
  = rect. AX, XB .... ab ....

Hence $AB^2 = AX^2 + XB^2 + 2AX \cdot XB;$

that is, $(a+b)^2 = a^2 + b^2 + 2ab.$

THEOREM 52. [Euclid II. 7.]

If a straight line is divided **externally** *at any point, the square on the given line is equal to the sum of the squares on the two segments* **diminished by** *twice the rectangle contained by the segments.*



Let AB be the given st. line divided *externally* at X; and let the segments AX, XB contain a and b units of length respectively.

Then AB is the *difference* of the segments AX, XB, and therefore contains $a-b$ units.

It is required to prove that

$$AB^2 = AX^2 + XB^2 - 2AX \cdot XB;$$

namely that $\quad (a-b)^2 = a^2 + b^2 - 2ab.$

Construction. On AX describe a square AXGE. From AE cut off AD equal to AB, or $a-b$. Then $ED = XB = b$. Through D and B draw DF, BH par[l] respectively to AX, AE, meeting at C.

Proof. Then the fig. AC = the figs. AG, CG − the figs. EF, XH.

And of these, by construction,

fig. AC is the sq. on AB, and contains $(a-b)^2$ units of area;

fig. AG = sq. on AX, and contains a^2 units of area;
fig. CG = sq. on XB, ... b^2 ....;
fig. EF = rect. EG, ED
= rect. AX, XB .... ab ....;
fig. XH = rect. GX, XB
= rect. AX, XB .... ab ....

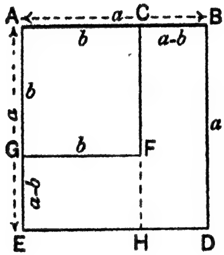
Hence $\quad AB^2 = AX^2 + XB^2 - 2AX \cdot XB;$

that is, $\quad (a-b)^2 = a^2 + b^2 - 2ab.$

Q.E.D.

THEOREM 53. [Euclid II. 5 and 6.]

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.



Let the given lines AB, AC be placed in the same st. line, and let them contain a and b units of length respectively.

It is required to prove that

$$AB^2 - AC^2 = (AB + AC)(AB - AC);$$

namely that $$a^2 - b^2 = (a+b)(a-b).$$

Construction. On AB and AC draw the squares ABDE, ACFG; and produce CF to meet ED at H.

Then $GE = CB = a - b$ units.

Proof. Now $AB^2 - AC^2$ = the sq. AD – the sq. AF
= the rect. CD + the rect. GH
= DB . BC + GF . GE
= AB . CB + AC . CB
= (AB + AC) CB
= (AB + AC)(AB – AC).

That is

$$a^2 - b^2 = (a+b)(a-b).$$

Q.E.D

COROLLARY. *If a straight line is bisected, and also divided (internally or externally) into two unequal segments, the rectangle contained by these segments is equal to the difference of the squares on half the line and on the line between the points of section.*

Fig. 1.

Fig. 2.

That is, if AB is bisected at X and also divided at Y, internally in Fig. 1, and externally in Fig. 2, then

in Fig. 1, $\quad AY \cdot YB = AX^2 - XY^2$;

in Fig. 2, $\quad AY \cdot YB = XY^2 - AX^2$.

For in the first case,

$$\begin{aligned} AY \cdot YB &= (AX + XY)(XB - XY) \\ &= (AX + XY)(AX - XY) \\ &= AX^2 - XY^2. \end{aligned}$$

The second case may be similarly proved.

EXERCISES.

1. Draw diagrams on squared paper to shew that the square on a straight line is

(i) *four*-times the square on *half* the line;

(ii) *nine*-times the square on *one-third* of the line.

2. Draw diagrams on squared paper to illustrate the following algebraical formulæ:

(i) $(x+7)^2 = x^2 + 14x + 49$.

(ii) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$.

(iii) $(a+b)(c+d) = ac + ad + bc + bd$.

(iv) $(x+7)(x+9) = x^2 + 16x + 63$.

3. In Theor. 50, Cor. (i), if AB = 4 cm., and the fig. AE = 9·6 sq. cm., find the area of the fig. XC.

4. In Theor. 50, Cor. (ii), if AX = 2·1″, and the fig. XC = 3·36 sq. in., find AB.

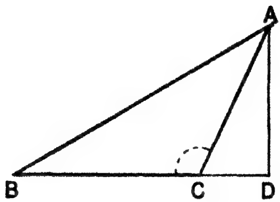
5. In Theor. 51, if the fig. AG = 36 sq. cm., and the rect. AX, XB = 24 sq. cm., find AB.

6. In Theorem 52, if the fig. AG = 9·61 sq. in., and the fig. DG = 6·51 sq. in., find AB.

[For further Examples on Theorems 50–53 see p. 230.]

Theorem 54. [Euclid II. 12.]

In an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle **together with** *twice the rectangle contained by one of those sides and the* **projection of the other side** *upon it.*



Let ABC be a triangle obtuse-angled at C; and let AD be drawn perp. to BC produced, so that CD is the projection of the side CA on BC. [See Def. p. 63.]

It is required to prove that

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$$

Proof. Because BD is the *sum* of the lines BC, CD,

$$\therefore \; BD^2 = BC^2 + CD^2 + 2BC \cdot CD. \qquad \textit{Theor. } 51.$$

To each of these equals add DA^2.

Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) + 2BC \cdot CD$.

$$\left.\begin{array}{l}\text{But } BD^2 + DA^2 = AB^2 \\ \text{and } CD^2 + DA^2 = CA^2\end{array}\right\}, \text{ for the } \angle D \text{ is a rt. } \angle.$$

Hence $\qquad AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$

Q.E.D.

THEOREM 55. [Euclid II. 13.]

In every triangle the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing that angle **diminished by** *twice the rectangle contained by one of those sides and the* **projection of the other side** *upon it.*

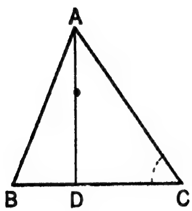


Fig. 1.

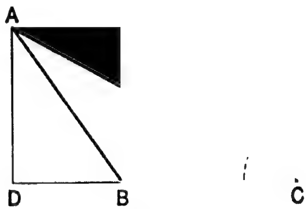


Fig. 2.

Let ABC be a triangle in which the $\angle$C is acute; and let AD be drawn perp. to BC, or BC produced; so that CD is the projection of the side CA on BC.

It is required to prove that

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$

Proof. Since in both figures BD is the *difference* of the lines BC, CD,

$$\therefore \quad BD^2 = BC^2 + CD^2 - 2BC \cdot CD. \qquad \textit{Theor. } 52.$$

To each of these equals add DA^2.

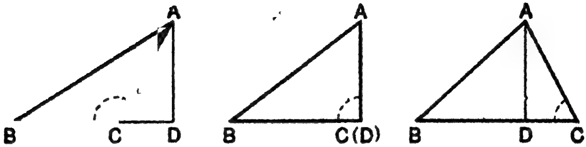
Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) - 2BC \cdot CD.$ ......(i)

$$\left.\begin{aligned} \text{But } BD^2 + DA^2 &= AB^2 \\ \text{and } CD^2 + DA^2 &= CA^2 \end{aligned}\right\}, \text{ for the } \angle D \text{ is a rt. } \angle.$$

Hence $AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$

Q.E.D

SUMMARY OF THEOREMS 29, 54 and 55.



(i) If the $\angle$ ACB is *obtuse*,

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$$ *Theor.* 54.

(ii) If the $\angle$ ACB is *a right angle*,

$$AB^2 = BC^2 + CA^2.$$ *Theor.* 29.

(iii) If the $\angle$ ACB is *acute*,

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$ *Theor.* 55.

Observe that in (ii), when the $\angle$ACB is *right*, AD coincides with AC, so that CD (the projection of CA) vanishes;

hence, in this case, $2BC \cdot CD = 0$.

Thus the three results may be collected in a single enunciation:

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other sides, according as the angle contained by those sides is obtuse, a right angle, or acute; the difference in cases of inequality being twice the rectangle contained by one of the two sides and the projection on it of the other.

EXERCISES.

1. In a triangle ABC, $a=21$ cm., $b=17$ cm., $c=10$ cm. By how many square centimetres does c^2 fall short of a^2+b^2? Hence or otherwise calculate the projection of AC on BC.

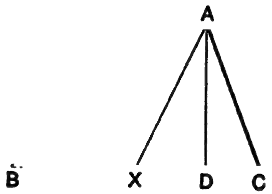
2. ABC is an isosceles triangle in which AB = AC; and BE is drawn perpendicular to AC. Shew that $BC^2 = 2AC \cdot CE$.

3. In the $\triangle$ ABC, shew that

(i) if the $\angle C = 60°$, then $c^2 = a^2 + b^2 - ab$;
(ii) if the $\angle C = 120°$, then $c^2 = a^2 + b^2 + ab$.

THEOREM 56.

In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.



Let ABC be a triangle, and AX the median which bisects the base BC.

It is required to prove that

$$AB^2 + AC^2 = 2BX^2 + 2AX^2.$$

Draw AD perp. to BC; and consider the case in which AB and AC are unequal, and AD falls within the triangle.

Then of the $\angle^s$ AXB, AXC, one is obtuse, and the other acute. Let the $\angle$ AXB be obtuse.

Then from the $\triangle$ AXB,

$$AB^2 = BX^2 + AX^2 + 2BX \cdot XD. \qquad \textit{Theor. 54.}$$

And from the $\triangle$ AXC,

$$AC^2 = XC^2 + AX^2 - 2XC \cdot XD. \qquad \textit{Theor. 55.}$$

Adding these results, and remembering that XC = BX, we have

$$AB^2 + AC^2 = 2BX^2 + 2AX^2.$$

Q.E.D.

NOTE. The proof may easily be adapted to the case in which the perpendicular AD falls outside the triangle.

EXERCISE.

In any triangle the difference of the squares on two sides is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertical angle to the base.

EXERCISES ON THEOREMS 50–53.

1. Use the Corollaries of Theorem 50 to shew that if a straight line AB is divided internally at X, then

$$AB^2 = AX^2 + XB^2 + 2AX \cdot XB.$$

2. If a straight line AB is bisected at X and produced to Y, and if $AY \cdot YB = 8AX^2$, shew that $AY = 2AB$.

3. *The sum of the squares on two straight lines is never less than twice the rectangle contained by the straight lines.*

Explain this statement by reference to the diagram of Theorem 52.

Also deduce it from the formula $(a-b)^2 = a^2 + b^2 - 2ab$.

4. In the formula $(a+b)(a-b) = a^2 - b^2$, substitute $a = \frac{x+y}{2}$, $b = \frac{x-y}{2}$, and enunciate verbally the resulting theorem.

5. If a straight line is divided internally at Y, shew that the rectangle AY, YB continually diminishes as Y moves from X, the mid-point of AB.

Deduce this (i) from the Corollary of Theorem 53;

(ii) from the formula $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$.

6. *If a straight line* AB *is bisected at* X, *and also divided* (i) *internally,* (ii) *externally into two unequal segments at* Y, *shew that in either case*

$$AY^2 + YB^2 = 2(AX^2 + XY^2). \qquad \text{[Euclid II. 9, 10.]}$$

[*Proof of case* (i).

$$\begin{aligned} AY^2 + YB^2 &= AB^2 - 2AY \cdot YB & \textit{Theor. } 51.\\ &= 4AX^2 - 2(AX + XY)(AX - XY) &\\ &= 4AX^2 - 2(AX^2 - XY^2) & \textit{Theor. } 53.\\ &= 2AX^2 + 2XY^2. & \end{aligned}$$

Case (ii) may be derived from Theorem 52 in a similar way.]

7. If AB is divided internally at Y, use the result of the last example to trace the changes in the value of $AY^2 + YB^2$, as Y moves from A to B.

8. *In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse.*

9. ABC is an isosceles triangle, and AY is drawn to cut the base BC internally or externally at Y. Prove that

$$AY^2 = AC^2 - BY \cdot YC, \text{ for internal section;}$$
$$AY^2 = AC^2 + BY \cdot YC, \text{ for external section.}$$

EXERCISES ON THEOREMS 54–56.

1. AB is a straight line 8 cm. in length, and from its middle point O as centre with radius 5 cm. a circle is drawn; if P is any point on the circumference, shew that

$$AP^2 + BP^2 = 82 \text{ sq. cm.}$$

2. In a triangle ABC, the base BC is bisected at X. If $a = 17$ cm., $b = 15$ cm., and $c = 8$ cm., calculate the length of the median AX, and deduce the $\angle A$.

3. The base of a triangle $= 10$ cm., and the sum of the squares on the other sides $= 122$ sq. cm.; find the locus of the vertex.

4. Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

The sides of a rhombus and its shorter diagonal each measure 3″; find the longer diagonal to within ·01″.

5. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [See Ex. 7, p. 64.]

6. ABCD is a rectangle, and O any point within it: shew that

$$OA^2 + OC^2 = OB^2 + OD^2.$$

If $AB = 6{\cdot}0''$, $BC = 2{\cdot}5''$, and $OA^2 + OC^2 = 21\tfrac{1}{4}$ sq. in., find the distance of O from the intersection of the diagonals.

7. The sum of the squares on the sides of a quadrilateral is greater than the sum of the squares on its diagonals by four times the square on the straight line which joins the middle points of the diagonals.

8. In a triangle ABC, the angles at B and C are acute; if BE, CF are drawn perpendicular to AC, AB respectively, prove that

$$BC^2 = AB \,.\, BF + AC \,.\, CE.$$

9. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

10. ABC is a triangle, and O the point of intersection of its medians: shew that

$$AB^2 + BC^2 + CA^2 = 3(OA^2 + OB^2 + OC^2).$$

11. If a straight line AB is bisected at X, and also divided (internally or externally) at Y, then

$$AY^2 + YB^2 = 2(AX^2 + XY^2).$$ [See p. 230 Ex. 6.]

Prove this from Theorem 56, by considering a triangle CAB in the limiting position when the vertex C falls at Y in the base AB.

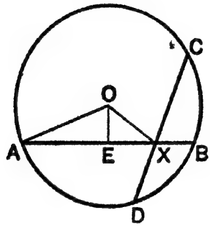
12. In a triangle ABC, if the base BC is divided at X so that $m\text{BX} = n\text{XC}$, shew that

$$m\text{AB}^2 + n\text{AC}^2 = m\text{BX}^2 + n\text{XC}^2 + (m+n)\text{AX}^2.$$

RECTANGLES IN CONNECTION WITH CIRCLES.

THEOREM 57. [Euclid III. 35.]

If two chords of a circle cut at a point within it, the rectangles contained by their segments are equal.



In the ⊙ ABC, let AB, CD be chords cutting at the internal point X.

It is required to prove that

the rect. AX, XB = *the rect.* CX, XD.

Let O be the centre, and r the radius, of the given circle.

Suppose OE drawn perp. to the chord AB, and therefore bisecting it.

Join OA, OX.

Proof. The rect. AX, XB = (AE + EX)(EB − EX)
= (AE + EX)(AE − EX)
= $AE^2 - EX^2$ *Theor.* 53.
= $(AE^2 + OE^2) - (EX^2 + OE^2)$
= r^2 − OX^2, since the ∠ˢ at E are rt. ∠ˢ.

Similarly it may be shewn that

the rect. CX, XD = $r^2 - OX^2$.

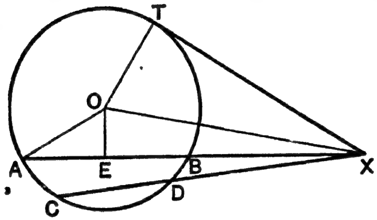
∴ the rect. AX, XB = the rect. CX, XD.

Q.E.D.

COROLLARY. *Each rectangle is equal to the square on half the chord which is bisected at the given point* X.

THEOREM 58. [Euclid III. 36.]

If two chords of a circle, when produced, cut at a point outside it, the rectangles contained by their segments are equal. And each rectangle is equal to the square on the tangent from the point of intersection.



In the ⊙ ABC, let AB, CD be chords cutting, when produced, at the external point X; and let XT be a tangent drawn from that point.

It is required to prove that

the rect. AX, XB = *the rect.* CX, XD = *the sq. on* XT.

Let O be the centre, and r the radius of the given circle.

Suppose OE drawn perp. to the chord AB, and therefore bisecting it.

Join OA, OX, OT.

Proof. The rect. AX, XB $= (EX + AE)(EX - EB)$

$= (EX + AE)(EX - AE)$

$= EX^2 - AE^2$ *Theor.* 53

$= (EX^2 + OE^2) - (AE^2 + OE^2)$

$OX^2 \quad - \quad r^2$, since the ∠s at E are rt. ∠s.

Similarly it may be shewn that

the rect. CX, XD $= OX^2 - r^2$.

And since the radius OT is perp. to the tangent XT,

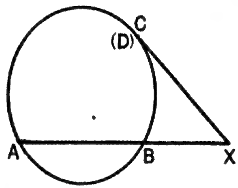
∴ $XT^2 = OX^2 - r^2$. *Theor.* 29.

∴ the rect. AX, XB = the rect CX, XD = the sq. on XT.

Q.E.D.

THEOREM 59. [Euclid III. 37.]

If from a point outside a circle two straight lines are drawn, one of which cuts the circle, and the other meets it; and if the rectangle contained by the whole line which cuts the circle and the part of it outside the circle is equal to the square on the line which meets the circle, then the line which meets the circle is a tangent to it.



From X a point outside the ⊙ ABC, let two straight lines XA, XC be drawn, of which XA cuts the circle at A and B, and XC meets it at C;

and let the rect. XA, XB = the sq. on XC.

It is required to prove that XC *touches the circle at* C.

Proof. Suppose XC meets the circle again at D;

then $XA \,.\, XB = XC \,.\, XD$. *Theor.* 58.

But by hypothesis, $XA \,.\, XB = XC^2$;

$\therefore\ XC \,.\, XD = XC^2$;

$\therefore\ XD = XC.$

Hence XC cannot meet the circle again unless the points of section coincide;

that is, XC is a tangent to the circle.

Q.E.D.

NOTE ON THEOREMS 57, 58.

Remembering that the segments into which the chord AB is divided at X, internally in Theorem 57, and externally in Theorem 58, are in each case AX, XB, we may include both Theorems in a single enunciation.

If any number of chords of a circle are drawn through a given point within or without a circle, the rectangles contained by the segments of the chords are equal.

EXERCISES ON THEOREMS 57–59.

(Numerical and Graphical.)

1. Draw a circle of radius 5 cm., and within it take a point X 3 cm. from the centre O. Through X draw any two chords AB, CD.

(i) Measure the segments of AB and CD; hence find approximately the areas of the rectangles AX . XB and CX . XD, and compare the results.

(ii) Draw the chord MN which is bisected at X; and from the right-angled triangle OXM calculate the value of XM^2.

(iii) Find by how much per cent. your estimate of the rect. AX, XB differs from its true value.

2. Draw a circle of radius 3 cm., and take an external point X 5 cm. from the centre O. Through X draw any two secants XAB, XCD.

(i) Measure XA, XB and XC, XD; hence find approximately the rectangles XA . XB and XC . XD, and compare the results.

(ii) Draw the tangent XT; and from the right-angled triangle XTO calculate the value of XT^2.

(iii) Find by how much per cent. your estimate of the rect. AX, XB differs from its true value.

3. AB, CD are two straight lines intersecting at X. AX = 1·8″, XB = 1·2″, and CX = 2·7″. If A, C, B, D are concyclic, find the length of XD.

Draw a circle through A, C, B, and check your result by measurement.

4. A secant XAB and a tangent XT are drawn to a circle from an external point X.

(i) If XA = 0·6″, and XB = 2·4″, find XT.

(ii) If XT = 7·5 cm., and XA = 4·5 cm., find XB.

5. A semi-circle is drawn on a given line AB; and from X, any point in AB, a perpendicular XM is drawn to AB cutting the circumference at M: shew that

$$AX . XB = MX^2.$$

(i) If AX = 2·5″, and MX = 2·0″, find XB; hence find the diameter of the semi-circle.

(ii) If the radius of the semi-circle = 3·7 cm., and AX = 4·9 cm., find MX.

6. A point X moves within a circle of radius 4 cm., and PQ is any chord passing through X; if in all positions PX . XQ = 12 sq. cm., find the locus of X.

What will the locus be if X moves outside the same circle, so that PX . XQ = 20 sq. cm.?

EXERCISES ON THEOREMS 57–59.

(Theoretical.)

1. ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse: shew that

$$AD . DB = CD^2.$$

2. If two circles intersect, and through any point X in their common chord two chords AB, CD are drawn, one in each circle, shew that

$$AX . XB = CX . XD.$$

3. Deduce from Theorem 58 that the tangents drawn to a circle from any external point are equal.

4. If two circles intersect, tangents drawn to them from any point in their common chord produced are equal.

5. If a common tangent PQ is drawn to two circles which cut at A and B, shew that AB produced bisects PQ.

6. If two straight lines AB, CD intersect at X so that AX . XB = CX . XD, deduce from Theorem 57 (by *reductio ad absurdum*) that the points A, B, C, D are concyclic.

7. In the triangle ABC, perpendiculars AP, BQ are drawn from A and B to the opposite sides, and intersect at O: shew that

$$AO . OP = BO . OQ.$$

8. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to the hypotenuse: shew that

$$AB . AD = AC^2.$$

9. Through A, a point of intersection of two circles, two straight lines CAE, DAF are drawn, each passing through a centre and terminated by the circumferences: shew that

$$CA . AE = DA . AF.$$

10. If from any external point P two tangents are drawn to a given circle whose centre is O and radius r; and if OP meets the chord of contact at Q; shew that

$$OP . OQ = r^2.$$

11. AB is a fixed diameter of a circle, and CD is perpendicular to AB (or AB produced); if *any* straight line is drawn from A to cut CD at P and the circle at Q, shew that

$$AP . AQ = \text{constant}.$$

12. A is a fixed point, and CD a fixed straight line; AP is *any* straight line drawn from A to meet CD at P; if in AP a point Q is taken so that AP . AQ is constant, find the locus of Q.

EXERCISES ON THEOREMS 57–59.

(*Miscellaneous.*)

1. The chord of an arc of a circle$=2c$, the height of the arc$=h$, the radius$=r$. Shew by Theorem 57 that

$$h(2r-h)=c^2.$$

Hence find the diameter of a circle in which a chord 24″ long cuts off a segment 8″ in height.

2. The radius of a circular arch is 25 feet, and its height is 18 feet: find the span of the arch.

If the height is reduced by 8 feet, the radius remaining the same, by how much will the span be reduced?

Check your calculated results graphically by a diagram in which 1″ represents 10 feet.

3. Employ the equation $h(2r-h)=c^2$ to find the height of an arc whose chord is 16 cm., and radius 17 cm.

Explain the double result geometrically.

4. If d denotes the shortest distance from an external point to a circle, and t the length of the tangent from the same point, shew by Theorem 58 that

$$d(d+2r)=t^2.$$

Hence find the diameter of the circle when $d=1{\cdot}2''$, and $t=2{\cdot}4''$; and verify your result graphically.

5. If the horizon visible to an observer on a cliff 330 feet above the sea-level is $22\frac{1}{2}$ miles distant, find roughly the diameter of the earth.

Hence find the approximate distance at which a bright light raised 66 feet above the sea is visible at the sea-level.

6. If h is the height of an arc of radius r, and b the chord of half the arc, prove that

$$b^2=2rh.$$

7. A semi-circle is described on AB as diameter, and any two chords AC, BD are drawn intersecting at P: shew that

$$AB^2=AC \,.\, AP+BD \,.\, BP.$$

8. Two circles intersect at B and C, and the two direct common tangents AE and DF are drawn: if the common chord is produced to meet the tangents at G and H, shew that

$$GH^2=AE^2+BC^2.$$

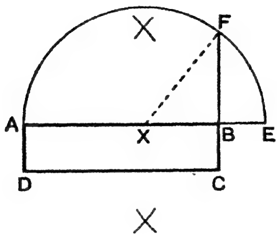
9. If from an external point P a secant PCD is drawn to a circle, and PM is perpendicular to a diameter AB, shew that

$$PM^2=PC \,.\, PD+AM \,.\, MB.$$

PROBLEMS.

PROBLEM 32.

To draw a square equal in area to a given rectangle.



Let ABCD be the given rectangle.

Construction. Produce AB to E, making BE equal to BC. On AE draw a semi-circle; and produce CB to meet the circumference at F.

Then BF is a side of the required square.

Proof. Let X be the mid-point of AE, and r the radius of the semi-circle. Join XF.

$$\begin{aligned} \text{Then the rect. } AC &= AB \cdot BE \\ &= (r + XB)(r - XB) \\ &= r^2 - XB^2 \\ &= FB^2, \text{ from the rt. angled } \triangle FBX. \end{aligned}$$

COROLLARY. *To describe a square equal in area to any given rectilineal figure.*

Reduce the given figure to a triangle of equal area. *Prob.* 18.

Draw a rectangle equivalent to this triangle. *Prob.* 17.

Apply to the rectangle the construction given above.

EXERCISES.

1. Draw a rectangle 8 cm. by 2 cm., and construct a square of equal area. What is the length of each side?

2. Find graphically the side of a square equal in area to a rectangle whose length and breadth are 3·0″ and 1·5″. Test your work by measurement and calculation.

3. Draw any rectangle whose area is 3·75 sq. in.; and construct a square of equal area. Find by measurement and calculation the length of each side.

4. Draw an equilateral triangle on a side of 3″, and construct a rectangle of equal area [Problem 17.] Hence find by construction and measurement the side of an equal square.

5. Draw a quadrilateral ABCD from the following data: $A=65°$; $AB=AD=9$ cm.; $BC=CD=5$ cm. Reduce this figure to a triangle [Problem 18], and hence to a rectangle of equal area. Construct an equal square, and measure the length of its side.

6. Divide AB, a line 9 cm. in length, internally at X, so that AX . XB = the square on a side of 4 cm.

Hence give a graphical solution, correct to the first decimal place, of the simultaneous equations:

$$x+y=9, \qquad xy=16.$$

7. Taking $\frac{1}{10}''$ as your unit of length, solve the following equations by a graphical construction, correct to one decimal figure:

$$x+y=40, \qquad xy=169.$$

8. The area of a rectangle is 25 sq. cm., and the length of one side is 7·2 cm.; find graphically the length of the other side to the nearest millimetre, and test your drawing by calculation.

9. Divide AB, a line 8 cm. in length, *externally* at X, so that AX . XB = the square on a side of 6 cm. [See p. 245.]

Hence find a graphical solution, correct to the first decimal place, of the equations:

$$x-y=8, \qquad xy=36.$$

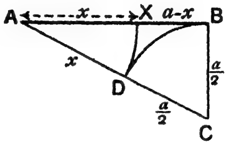
10. On a straight line AB draw a semi-circle, and from any point P on the circumference draw PX perpendicular to AB. Join AP, PB, and denote these lines by x and y.

Noticing that (i) $x^2+y^2=AB^2$; (ii) $xy=2\triangle APB=AB \cdot PX$; devise a graphical solution of the equations:

$$x^2+y^2=100, \qquad xy=25.$$

Problem 33.

To divide a given straight line so that the rectangle contained by the whole and one part may be equal to the square on the other part.



Let AB be the st. line to be divided at a point X in such a way that

$$AB \cdot BX = AX^2.$$

Construction. Draw BC perp. to AB, and make BC equal to half AB. Join AC.

From CA cut off CD equal to CB.
From AB cut off AX equal to AD.
Then AB is divided as required at X.

Proof. Let $AB = a$ units of length, and let $AX = x$.

Then $BX = a - x$; $AD = x$; $BC = CD = \frac{a}{2}$.

Now $AB^2 = AC^2 - BC^2$, from the rt. angled $\triangle ABC$:
$= (AC - BC)(AC + BC)$;

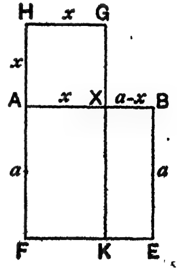
that is, $a^2 = x(x+a)$
$= x^2 + ax.$

From each of these equals take ax;

then $a^2 - ax = x^2$;

or, $a(a-x) = x^2$,

that is, $AB \cdot BX = AX^2$.

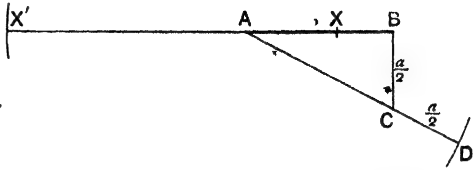


EXERCISE.

Let AB be divided as above at X. On AB, AX, and on opposite sides of AB, draw the squares ABEF, AXGH; and produce GX to meet FE at K. In this diagram name rectangular figures equivalent to a^2, x^2, $x(x+a)$, ax, and $a(a-x)$. Hence illustrate the above proof graphically.

NOTE. A straight line is said to be divided in **Medial Section** when the rectangle contained by the given line and one segment is equal to the square on the other segment.

This division may be *internal* or *external*; that is to say, AB may be divided internally at X, and externally at X′, so that



(i) $AB.BX=AX^2$,
(ii) $AB.BX'=AX'^2$.

To obtain X′, the construction of p. 240 must be modified thus:

CD is to be cut off from AC *produced*;

AX′.................... from BA *produced*, in the *negative* sense.

ALGEBRAICAL ILLUSTRATION.

If a st. line AB is divided at X, internally or externally, so that

$$AB.BX=AX^2,$$

and if $AB=a$, $AX=x$, and consequently $BX=a-x$, then

$$a(a-x)=x^2,$$

or,

$$x^2+ax-a^2=0,$$

and the roots of this quadratic, namely, $\frac{a\sqrt{5}}{2}-\frac{a}{2}$ and $-\left(\frac{a\sqrt{5}}{2}+\frac{a}{2}\right)$, are the lengths of AX and AX′.

EXERCISES.

1. Divide a straight line 4″ long internally in medial section. Measure the greater segment, and find its length algebraically.

2. Divide AB, a line 2″ long, externally in medial section at X′. Measure AX′, and obtain its length algebraically, explaining the geometrical meaning of the negative sign.

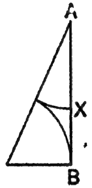
3. In the figure of Problem 33, shew that $AC=\frac{a\sqrt{5}}{2}$. [Theor. 29.]

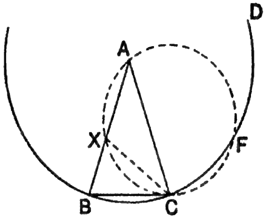
Hence prove (i) $AX=\frac{a\sqrt{5}}{2}-\frac{a}{2}$; (ii) $AX'=-\left(\frac{a\sqrt{5}}{2}+\frac{a}{2}\right)$.

4. If a straight line is divided internally in medial section, and from the greater segment a part is taken equal to the less, shew that the greater segment is also divided in medial section.

PROBLEM 34.

To draw an isosceles triangle having each of the angles at the base double of the vertical angle.





Construction. Take any line AB, and divide it at X, so that $AB \cdot BX = AX^2$. *Prob.* 33.

(This construction is shewn separately on the left.)

With centre A, and radius AB, draw the ⊙ BCD; and in it place the chord BC equal to AX.

Join AC.

Then ABC is the triangle required.

Proof. Join XC, and suppose a circle drawn through A, X and C.

Now, by construction, $BA \cdot BX = AX^2$
$= BC^2$;

∴ BC touches the ⊙ AXC at C; *Theor.* 59.

∴ the $\angle BCX$ = the $\angle XAC$, in the alt. segment.

To each add the $\angle XCA$;

then the $\angle BCA$ = the $\angle XAC$ + the $\angle XCA$
= the ext. $\angle CXB$.

And the $\angle BCA$ = the $\angle CBA$, for AB = AC.

∴ the $\angle CBX$ = the $\angle CXB$;

∴ CX = CB = AX,

∴ the $\angle XAC$ = the $\angle XCA$;

∴ the $\angle XAC$ + the $\angle XCA$ = twice the $\angle A$.

But the $\angle^s ABC$ = the $\angle ACB$ = the $\angle XAC$ + the $\angle XCA$ *Proved.*
= twice the $\angle A$.

EXERCISES.

1. How many degrees are there in the vertical angle of an isosceles triangle in which each angle at the base is double of the vertical angle?

2. Shew how a right angle may be divided into five equal parts by means of Problem 34.

3. In the figure of Problem 34 point out a triangle whose vertical angle is three times either angle at the base.

Shew how such a triangle may be constructed.

4. If in the triangle ABC, the $\angle B =$ the $\angle C =$ twice the $\angle A$, shew that

$$\frac{BC}{AB} = \frac{\sqrt{5}-1}{2}.$$

5. In the figure of Problem 34, if the two circles intersect at F, shew that

(i) $BC = CF$;

(ii) the circle $AXC =$ the circum-circle of the triangle ABC;

(iii) BC, CF are sides of a regular decagon inscribed in the circle BCD;

(iv) AX, XC, CF are sides of a regular pentagon inscribed in the circle AXC.

6. In the figure of Problem 34, shew that the centre of the circle circumscribed about the triangle CBX is the middle point of the arc XC.

7. In the figure of Problem 34, if I is the in-centre of the triangle ABC, and I′, S′ the in-centre and circum-centre of the triangle CBX, shew that $S'I = S'I'$.

8. If a straight line is divided in medial section, the rectangle contained by the sum and difference of the segments is equal to the rectangle contained by the segments.

9. If a straight line AB is divided internally in medial section at X, shew that

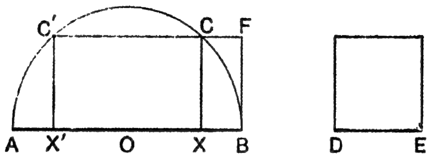
$$AB^2 + BX^2 = 3AX^2.$$

Also verify this result by substituting the values given on page 241.

THE GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS.

From the following constructions, which depend on Problem 32, a graphical solution of easy quadratic equations may be obtained.

I. *To divide a straight line* **internally** *so that the rectangle contained by the segments may be equal to a given square.*



Let AB be the st. line to be divided, and DE a side of the given square.

Construction. On AB draw a semicircle; and from B draw BF perp. to AB and equal to DE.

From F draw FCC′ par[l] to AB, cutting the $\bigcirc^{ce}$ at C and C′.

From C, C′ draw CX, C′X′ perp. to AB.

Then AB is divided as required at X, and also at X′.

Proof. $AX \cdot XB = CX^2$ *Prob.* 32.

$= BF^2$

$= DE^2$.

Similarly $AX' \cdot X'B = DE^2$.

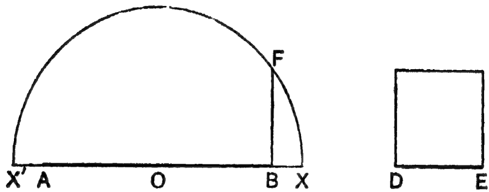
Application. The purpose of this construction is to find two straight lines AX, XB, having given their *sum*, viz. AB, and their *product*, viz. the square on DE.

Now to solve the equation $x^2 - 13x + 36 = 0$, we have to find two numbers whose *sum* is 13, and whose *product* is 36, or 6^2.

To do this graphically, perform the above construction, making AB equal to 13 cm., and DE equal to $\sqrt{36}$ or 6 cm. The segments AX, XB represent the roots of the equation, and their values may be obtained by measurement.

NOTE. If the last term of the equation is not a perfect square, as in $x^2 - 7x + 11 = 0$, $\sqrt{11}$ must be first got by the arithmetical rule, or graphically by means of Problem 32.

II. *To divide a straight line* **externally** *so that the rectangle contained by the segments may be equal to a given square.*



Let AB be the st. line to be divided externally, and DE the side of the given square.

Construction. From B draw BF perp. to AB, and equal to DE. Bisect AB at O.

With centre O, and radius OF draw a semi-circle to cut AB produced at X and X′.

Then AB is divided externally as required at X, and also at X′.

Proof. $AX \,.\, XB = X'B \,.\, BX$, since $AX = X'B$,

$= BF^2$ *Prob.* 32.

$= DE^2$.

Application. Here we find two lines AX, XB, having given their *difference*, viz. AB, and their *product*, viz. the square on DE.

Now to solve the equation $x^2 - 6x - 16 = 0$, we have to find two numbers whose numerical *difference* is 6, and whose product is 16, or 4^2.

To do this graphically, perform the above construction, making AB equal to 6 cm., and DE equal to $\sqrt{16}$ or 4 cm. The segments AX, XB represent the roots of the equation, and their values, as before, may be obtained by measurement.

EXERCISES.

Obtain approximately the roots of the following quadratics by means of graphical constructions; and test your results algebraically.

1. $x^2 - 10x + 16 = 0$. 2. $x^2 - 14x + 49 = 0$. 3. $x^2 - 12x + 25 = 0$.
4. $x^2 - 5x - 36 = 0$. 5. $x^2 - 7x - 49 = 0$. 6. $x^2 - 10x + 20 = 0$.

EXERCISES FOR SQUARED PAPER.

1. A circle passing through the points (0, 4), (0, 9) touches the x-axis at P. Calculate and measure the length of OP.

2. With centre at the point (9, 6) a circle is drawn to touch the y-axis. Find the rectangle of the segments of any chord through O. Also find the rectangle of the segments of any chord through the point (9, 12).

3. Draw a circle (shewing all lines of construction) through the points (6, 0), (24, 0), (0, 9). Find the length of the other intercept on the y-axis, and verify by measurement. Also find the length of a tangent to the circle from the origin.

4. Draw a circle through the points (10, 0), (0, 5), (0, 20); and prove by Theorem 59 that it touches the x-axis.

Find (i) the coordinates of the centre, (ii) the length of the radius.

5. If a circle passes through the points (16, 0), (18, 0), (0, 12), shew by Theorem 58 that it also passes through (0, 24).

Find (i) the coordinates of the centre, (ii) the length of the tangent from the origin.

6. Plot the points A, B, C, D from the coordinates (12, 0), (−6, 0), (0, 9), (0, −8); and prove by Theorem 57 that they are concyclic.

If r denotes the radius of the circle, shew that

$$OA^2+OB^2+OC^2+OD^2=4r^2.$$

7. Draw a circle (shewing all lines of construction) to touch the y-axis at the point (0, 9), and to cut the x-axis at (3, 0).

Prove that the circle must cut the x-axis again at the point (27, 0); and find its radius. Verify your results by measurement.

8. Shew that two circles of radius 13 may be drawn through the point (0, 8) to touch the x-axis; and by means of Theorem 58 find the length of their common chord.

9. Given a circle of radius 15, the centre being at the origin, shew how to draw a second circle of the same radius touching the given circle and also touching the x-axis.

How many circles can be so drawn? Measure the coordinates of the centre of that in the first quadrant.

10. A, B, C, D are four points on the x-axis at distances 6, 9, 15, 25 from the origin O. Draw two intersecting circles, one through A, B, and the other through C, D, and hence determine a point P in the x-axis such that

$$PA \,.\, PB=PC \,.\, PD.$$

Calculate and measure OP.

If the distances of A, B, C, D from O are a, b, c, d respectively, prove that

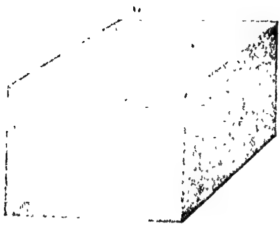
$$OP=(ab-cd)/(a+b-c-d).$$

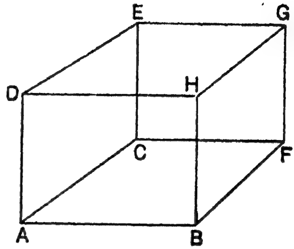
APPENDIX.

APPENDIX.

ON THE FORM OF SOME SOLID FIGURES.

(*Rectangular Blocks.*)





The solid whose shape you are probably most familiar with is that represented by a brick or slab of hewn stone. This solid is called a **rectangular block** or **cuboid.** Let us examine its form more closely.

How many *faces* has it? How many *edges*? How many *corners*, or *vertices*?

The faces are quadrilaterals: of what shape?

Compare two opposite faces. Are they equal? Are they parallel?

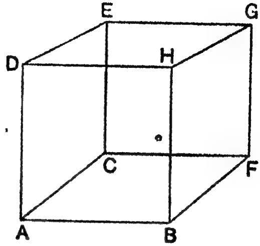
We may now sum up our observations thus:

A cuboid has *six* faces; opposite faces being *equal rectangles in parallel planes.* It has *twelve* edges, which fall into three groups, corresponding to the *length*, the *breadth*, and the *height* of the block. The four edges in each group are equal and parallel, and perpendicular to the two faces which they cut.

The length, breadth, and height of a rectangular block are called its **three dimensions.**

Ex. 1. If two dimensions of a rectangular block are equal, say, the breadth AC and the height AD, two faces take a particular shape. Which faces? What shape?

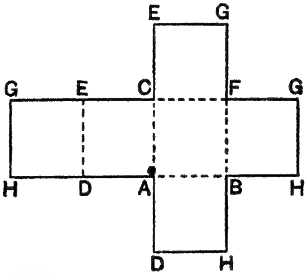
Ex. 2. If the length, breadth, and height of a rectangular block are all equal, what shapes do the faces take?



A rectangular block whose length, breadth, and height are all equal is called a **cube**. Its surface consists of six equal squares.

We will now see how models of these solids may be constructed, beginning with the cube, as being the simpler figure.

Suppose the surface of the cube to be cut along the upright edges, and also along the edge HG; and suppose the faces to be unfolded and flattened out on the plane of the base. The surface would then be represented by a figure consisting of six squares arranged as below.



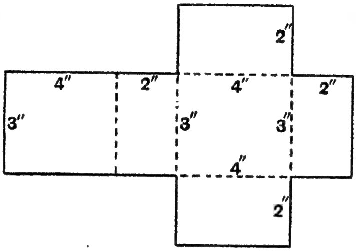
This figure is called the **net** of the cube: it is here drawn on half the scale of the cube shewn in outline above.

To make a model of a cube, draw its net on cardboard. Cut out the net along the outside lines, and cut partly through along the dotted lines. Fold the faces over till the edges come together; then fix the edges in position by strips of gummed paper.

Ex. 3. Make a model of a cube each of whose edges is 6·0 cm.

Ex. 4. Make a model of a rectangular block, whose length is 4″, breadth 3″, height 2″.

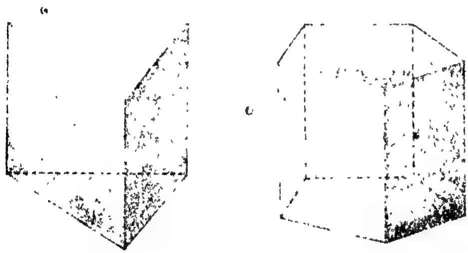
First draw the net which will consist of six rectangles arranged as below, and having the dimensions marked in the diagram.



Now cut the net out, fold the faces along the dotted lines, and secure the edges with gummed paper, as already explained.

(*Prisms.*)

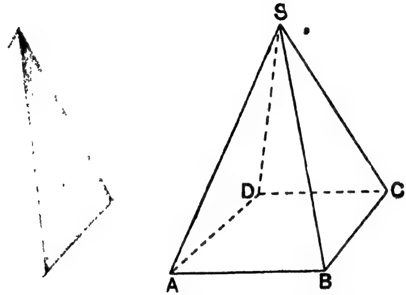
Let us now consider a solid whose side-faces (as in a rectangular block) are rectangles, but whose *ends* (*i.e.* base and top), though equal and parallel, are not necessarily *rectangles.* Such a solid is called a **prism.**



The ends of a prism may be any congruent figures: these may be triangles, quadrilaterals, or polygons of any number of sides. The diagram represents two prisms, one on a triangular base, the other on a pentagonal base.

Ex. 5. Draw the net of a triangular prism, whose ends are equilateral triangles on sides of 5 cm., and whose side-edges measure 7 cm.

(Pyramids.)



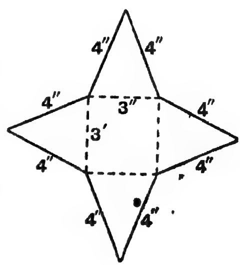
The solid represented in this diagram is called a **pyramid.**

The base of a pyramid (as of a prism) may have any number of sides, but the side-faces must be *triangles* whose vertices are at the same point.

The particular pyramid shewn in the Figure stands on a *square* base ABCD, and its side-edges SA, SB, SC, SD are all equal. In this case the side-faces are equal isosceles triangles; and the pyramid is said to be *right*, for if the base is placed on a level table, then the vertex lies in an upright line through the mid-point of the base.

Ex. 6. Make a model of a right pyramid standing on a square base. Each edge of the base is to measure 3″, and each side-edge of the pyramid is to be 4″.

To make the necessary net, draw a square on a side of 3″. This will form the base of the pyramid. Then on the sides of this square draw isosceles triangles making the equal sides in each triangle 4″ long.



Explain why the process of folding about the dotted lines brings the four vertices together.

Another important form of pyramid has as base an equilateral triangle, and all the side edges are equal to the edges of the base.

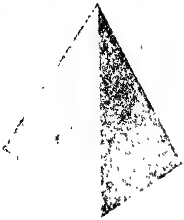


FIG. 1.

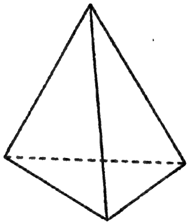


FIG. 2.

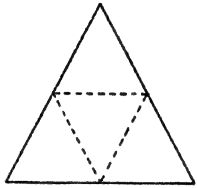


FIG. 3.

How many faces will such a pyramid have? How many edges? What sort of triangles will the side-faces be? Fig. 3 shews the net on a reduced scale.

A pyramid of this kind is called a regular **tetrahedron** (from Greek words meaning *four-faced*).

Ex. 7. Construct a model of a regular tetrahedron, each edge of which is 3″ long.

Ex. 8. What is the smallest number of *plane* faces that will enclose a space? What is the smallest number of *curved* surfaces that will enclose a space?

(*Cylinders.*)

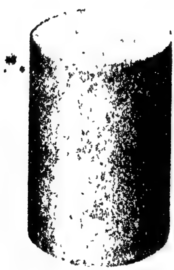


FIG. 1.

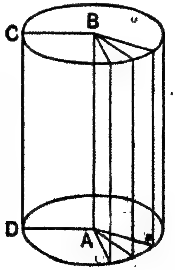


FIG. 2.

The solid figure here represented is called a **cylinder**.

On examining the model of which the last diagram is a drawing, you will notice that the two ends are *plane, circular, equal,* and *parallel.*

The side-surface is curved, but not curved in every direction; for it is evidently possible in one direction to rule *straight* lines on the surface: in *what* direction?

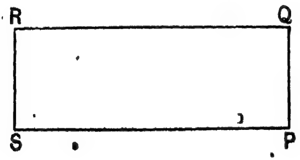
Let us take a rectangle ABCD (see Fig. 2), and suppose it to rotate about one side AB as a fixed axis.

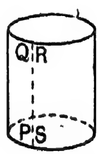
What will BC and AD trace out, as they revolve about AB?

Observe that CD will move so as always to be parallel to the axis AB, and to pass round the curve traced out by D. As CD moves, it will generate (that is to say, *trace out*) a surface. What sort of surface?

We now see why in *one* direction, namely parallel to the axis AB, it is possible to rule *straight* lines on the *curved* surface of a cylinder.

It is easy to find a plane surface to represent the curved surface of a cylinder.





Cut a rectangular strip of paper, making the width PQ equal to the height of the cylinder. Wrap the paper round the cylinder, and carefully mark off the length PS that will make the paper go exactly once round. Cut off all that overlaps; and then unwrap the covering strip. You have now a rectangle representing the curved surface of the cylinder, and having the same area.

(Cones.)

FIG. 1.

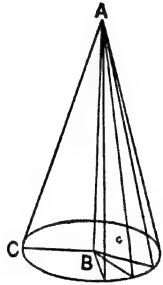


FIG. 2.

We have now to examine the model of a **cone**, of which a drawing is given above.

Its surface consists of two parts; first a *plane circular* base, then a *curved surface* which tapers from the circumference of the base to a point above it called the vertex. Thus the form of a cone suggests a pyramid standing on a circular instead of a rectilineal base.

Let us take a triangle ABC right-angled at B (Fig. 2), and suppose it to rotate about one side AB as a fixed axis. What will BC trace out as the triangle revolves? Notice that AC will always pass through the *fixed* point A, and move round the curve traced out by C. As AC moves, it will generate a surface. What sort of surface?

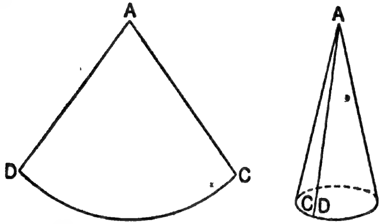
We now see that the kind of cone represented in the diagram is a solid generated by the revolution of a right-angled triangle about one side containing the right angle.

Ex. 9. Why must the $\triangle$ ABC, rotating about AB, be *right-angled* at B, in order to generate a cone?

What would be generated by the revolution of an *obtuse*-angled triangle about one side forming the obtuse angle?

Ex. 10. What would be generated by an *oblique* parallelogram revolving about one side?

The curved surface of a cone may be represented by a plane figure thus:



Taking the slant-height AC of the cone as radius, draw a circle. Cut it out from your paper; call its centre A; and cut it along any radius AC. If you now place the centre of the circular paper at the vertex of the cone, you will find that you can wrap the paper round the cone without fold or crease. Mark off from the circumference of your paper the length CD that will go exactly once round the base of the cone; then cut through the radius AD. We have now a plane figure ACD (called a *sector of a circle*) which represents the curved surface of the cone, and has the same area.

(*Spheres.*)

The last solid we have to consider is the **sphere**, whose shape is that of a globe or billiard ball.

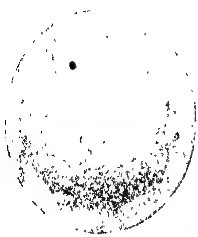


FIG. 1.

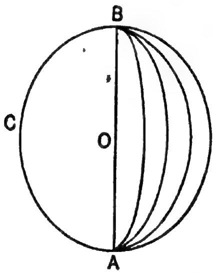


FIG. 2.

We shall realise its form more definitely, if we imagine a semi-circle ACB (Fig. 2) to rotate about its diameter as a fixed axis. Then, as the semi-circumference revolves, it generates the surface of a sphere.

Now since all points on the semi-circumference are in all positions at a constant distance from its centre O, we see that all points on the surface of a sphere are at a constant distance from a fixed point within it, namely the centre. This constant distance is the radius of the sphere. Thus all straight lines through the centre terminated both ways by the surface are equal: such lines are *diameters*.

Ex. 11. We have seen that on the curved surfaces of a cylinder and cone it is possible (in certain ways only) to rule *straight* lines. Is there any direction in which we can rule a straight line on the surface of a sphere?

Ex. 12. Again we have cut out a *plane* figure that could be wrapped round the *curved* surface of a cylinder without folding, creasing, or stretching. The same has been done for the curved surface of a cone. Can a flat piece of paper be wrapped about a sphere so as to fit all over the surface without creasing?

Ex. 13. Suppose you were to cut a sphere straight through the centre into two parts, in such a way that the new surfaces (made by cutting) are *plane*, these parts would be in every way alike. The parts into which a sphere is divided by a *plane central section* are called hemispheres. Of what shape is the line in which the plane surface meets the curved surface? If the section were *plane* but not *central*, can you tell what the meeting line of the two surfaces would be?

Ex. 14. If a cylinder were cut by a plane parallel to the base, of what shape would the new rim be?

Ex. 15. If a cone were cut by a plane parallel to the base, what would be the form of the section?

ANSWERS TO NUMERICAL EXERCISES.

Since the utmost care cannot ensure absolute accuracy in graphical work, results so obtained are likely to be only approximate. The answers here given are those found by calculation, and being true so far as they go, furnish a standard by which the student may test the correctness of his drawing and measurement. Results within one per cent. of those given in the Answers may usually be considered satisfactory.

Exercises. Page 15.

1. 30°; 126°; 261°; 85°. 11 min.; 37 min.
2. $112\frac{1}{2}$°; 155°; 5 hrs. 45 min. **3.** 50°; 8 hrs. 40 min.
4. (i) 145°, 35°, 145°. (ii) 55°, 55°. 86°, 94°.

Exercises. Page 27.

1. 68°, 37°, 75° v. nearly. **2.** 6·0 cm. **4.** 2·2′, 50°, 73° nearly.
5. 37 ft. **6.** 101 metres. **7.** 27 ft. **8.** 424 yds., nearly; N.W.
9. 281 yds.; 155 yds., 153 yds. **10.** 214 yds.

Exercises. Page 41.

1. 125°, 55°, 125°. **12.** 15 secs., 30 secs.

Exercises. Page 43.

3. 21°. **4.** 27°. **5.** 92°, 46°. **6.** 67°, 62°.

Exercises. Page 45.

1. 30°, 60°, 90°. **2.** (i) 36°, 72°, 72°; (ii) 20°, 80°, 80°.
3. 40°. **4.** 51°, 111°, 18°. **5.** (i) 34°; (ii) 107°.
6. 68°. **7.** 120°. **8.** 36°, 72°, 108°, 144°.
9. 165°. **11.** 5, 15.

Exercises. Page 47.

2. (i) 45°; (ii) 36°. **3.** (i) 12; (ii) 15.

Exercises. Page 54.

4. (i) 81°; *c.* (ii) 55°.

10.

Degrees	15°	30°	45°	60°	75°
Cm.	4·1	4·6	5·7	8·0	15·6

11.

Degrees	0°	30°	60°	90°	120°	150°	180°
Cm.	1·0	2·0	3·6	5·0	6·1	6·8	7·0

12. 37 ft. **13.** 112 ft. **14.** 346 yds. 693 yds.

Exercises. Page 61.

14. 54°, 72°, 54°. 15. 36°. 16. 4.
18. (i) 16; (ii) 45°; (iii) $11\frac{1}{4}$° per sec.

Exercises. Page 68.

2. 6·80 cm. 3. 2·24″. 4. 0·39. 5. 2·54. 8. 10·6 cm.
9. 3·35″. 10. 20 miles; 12·6 km.
11. 147 miles; 235 km. 1 cm. represents 22 km.
12. 1″ represents 15 mi.; 1″ represents 20 mi.

Exercises. Page 79.

3. 0·53 in. 4. 1·3 cm. 5. 2·4″.

Exercises. Page 84.

1. 4·3 cm., 5·2 cm., 6·1 cm. 2. 1·10. 3. 200 yards.
4. 65°, 77 m., 61 m., 56 m. 5. 6·04 knots. S, 15° E, nearly.
6. Results equal. 9 cm. 7. 4·3 cm.; 9·8 cm., 60°; 120°.
8. (i) One solution; (ii) two; (iii) one, right-angled; (iv) impossible.
9. 380 yds. 10. 6·5 cm. 11. 6·9 cm.
12. Two solutions; 10·4 cm. or 4·5 cm. 16. 2·8 cm., 4·5 cm., 5·3 cm.
18. 5·8 cm., 4·2 cm. 19. 7 cm., 8 cm.

Exercises. Page 89.

1. 60°, 120°. 2. 3·54″. 3. 2·12″. 4. 4·4 cm.
5. 16·4 cm., 3·4%. 6. 90°. 7. (i) 4·25″; (ii) $B=D=90°$.

Exercises. Page 102.

1. 6 sq. in. 2. 6 sq. in. 3. 2·80 sq. in. 4. 3·50 sq. in.
5. 3·30 sq. in. 6. 3·36 sq. in. 7. 198 sq. m. 8. 42 sq. ft.
9 10,000 sq. m. 10. 110 sq. ft. 11. 5 cm. 12. 2·6 in.
14. 900 sq. yds.; 48 yds.; 4·8″. 15. 11700 sq. m.
16. 1 cm. = 10 yds. 17. 1·6″. 18. 600 sq. ft. 19. 1152 sq. ft.
20. 100 sq. ft. 21. 156 sq. ft. 22. 110 sq. ft.
23. 288 sq. ft. 24. 72 sq. ft. 25. 75 sq. ft.

Exercises. Page 105.

1. (i) 22 cm.; (ii) 3·6″. 2. 3·4 sq. in. 3. 574·5 sq. in.
4. 1·5″. 5. 1·93″, 75°.

Exercises. Page 107.

1. (i) 180 sq. ft.; (ii) 8·4 sq. in.; 1 hectare.
2. (i) 13·44 sq. cm.; (ii) 15·40 sq. cm.; (iii) 20·50 sq. cm.
3. 15 sq. cm. 4. 6·3 sq. in.
5. (i) 8″; (ii) 13 cm. 6. 3·36 sq. in.

Exercises. Page 110.

1. 11400 sq. yds. 2. 6312 sq. m.
3. 2·4 cm.; 5·1 cm. 4. 2·04″; 2·20″.

5.

Angle	0°	30°	60°	90°	120°	150°	180°
Area in sq. cm.	0	7·5	13·0	15·0	13·0	7·5	0

Exercises. Page 111.

1. 66 sq. ft. 2. 84 sq. yds. 3. 126 sq. m.
4. 132 sq. cm. 5. 180 sq. ft. 6. 306 sq. m.

Exercises. Page 113.

1. 6 sq. in. 2. 170 sq. ft. 3. 615 sq. m. 4. 8·4 sq. in.
5. 31·2 sq. cm. 6. 5·20 sq. in. 7. 24 sq. cm.

Exercises. Page 115.

1. (i) 25·5 sq. cm.; (ii) 15·6 sq. cm.
2. (i) 8·95 sq. in.; (ii) 9·5 sq. in. 3. 12500 sq. m.

Exercises. Page 116.

4. 3·3 sq. in. 5. 7·5 cm. 6. 3·6 sq. in.

Exercises. Page 121.

1. (i) 5 cm.; (ii) 6·5 cm.; (iii) 3·7″. 2. (i) 1·6″; (ii) 2·8 cm.
3. 41 ft. 4. 65 miles. 5. 6·1 km. 6. 16 ft.
7. 48 m. 8. 25 miles. 9. 73 m. 10. 62 ft.

Exercises. Page 123.

10. (i) and (iii). 11. 2·83″. 12. 4·24 cm. ; 18 sq. cm.
13. 70·71 sq. m. 14. $p=6{\cdot}93$ cm.
16. (i) 20 cm. ; 15 cm. (ii) 40 cm. ; 39 cm.
17. 35 cm. ; 12 cm. ; 306 sq. cm.
18. (i) 36 sq. in. ; (ii) 90 sq. ft. ; (iii) 126 sq. cm. ; (iv) 240 sq. yds.
19. 5·1 cm. nearly.

Exercises. Page 127.

1. 7·1 cm. 4. 4·0 cm. 5. 1·6″. 6. 3·1 cm. ; 15·6 sq. cm.

Exercises. Page 130.

1. 23·90 sq. cm. 2. 8·40 sq. in.
3. 27·52 sq. cm. 4. 129800 sq. m.

Exercises. Page 134.

3. (i) (8, 5) ; (ii) (10, 10).
4. (i) (4, 5) ; (ii) (4, 5) ; (iii) (−4, −5) ; (iv) (−4, −5).
5. (6, 5), (12, 10). 6. (5, 8).
7. (i) 17 ; (ii) 17 ; (iii) 2·5″ ; 2·5″.
8. (i) and (ii) 5 ; (iii) and (iv) 17 ; (v) and (vi) 37. 9. 10.
14. (0, 0). (7, 5). 15. 13 ; (9, 6).
16. A straight line passing through the points (4, 0), (0, −4).
17. 117 units of area in each case. 18. A square. 2 sq. in. 1 sq. in.
19. Each = 70 units of area. 20. 9 units of area. 31°, 71°, 78°.
21. (i) 96 ; (ii) 80 ; (iii) 120 ; (iv) 104.
22. (i) 50 ; (ii) 60 ; (iii) 120 ; (iv) 132.
23. Sides 5, 13 ; area 63. 24. (i) 27 ; (ii) 21 ; (iii) 30 ; (iv) 27·5.
25. (i) 50 ; (ii) 65·5 ; (iii) 21 ; (iv) 83·5.
26. Each side 13 ; area 120. * 27. 13, 10, 15, 8·24, 42, 30.
28. AB = 10, BC = 9, CD = 17, DA = 12·7. Area = 130·5.
29. 10, 13, 5, 5, 3. Area = 60. 30. 160,000 sq. yds. 1000 yds. 320 yds.
31. Side = 15·23 ; area = 232 units of area.

Exercises. Page 145.

1. 5 cm. 2. 24″. 3. 0·6″, 0·8″. 4. $\sqrt{7}=2{\cdot}6$ cm.
5. 1 ft. 6. 0·6 sq. in. 7. 0·8″.

Exercises. Page 149.

1. 1·7″. 2 $3\sqrt{2}=4\cdot2$ cm. 3. $2\sqrt{3}=3\cdot5$ cm.
4. 17″. 6. 5 cm.

Exercises. Page 151.

6. 4 cm. 7. 1·3″.

Exercises. Page 153.

2. 1·85″. 3. 1·62″. 5. 0·85″; (2·1″, 2·1″); 2·97″.

Exercises. Page 155.

5. 51″. 6. 1·6″; 1·5″, 0·6″.

Exercises. Page 157.

4. (8, 11). 5. 17; 10; (0, −8).

Exercises. Page 161.

1. 74°, 148°, 16°. 2. 115°, 230°. 3. 55°, 8°, 47°.

Exercises. Page 177.

1. 8·0 cm. 2. 0·6″. 3. 8·7 cm. 4. 12″, 67°. 5. 2·5″.

Exercises. Page 179.

3. 3 cm. and 17 cm.

Exercises. Page 181.

1. 72°, 108°, 108°.

Exercises. Page 187.

2. 1·6″. 3. 1·7″. 4. 1·98″, 1·6″.

Exercises. Page 198.

2. 2·3 cm., 4·6 cm., 6·9 cm. 3. 1·39″.
6·9 cm.; 20·78 sq. cm. 7. 3·2 cm.

Exercises. Page 199

1. 2·12″; 4·50 sq. in. 4. 8·5 cm. 5. 2·0″.

Exercises. Page 200.

4. $128\frac{4}{7}$°; 1·73″.

Exercises. Page 201.

1. 3·46″; 4·00″. 2. 259·8 sq. cm.
4. (i) 41·57 sq. cm.; (ii) 77·25 sq. cm.

Exercises. Page 205.

1. (i) 28·3 cm.; (ii) 628·3 cm. 2. (i) 16·62 sq. in.; (ii) 352·99 sq. in.
3. 11·31 cm.; 10·18 sq. cm. 4. 56 sq. cm. 5. 43·98 sq. in.
7. 30·5 sq. cm. 8. 8·9″. 9. 4″; 3″. 10. 12·57 sq. in.
11. Circumferences, 4·4″, 6·3″. Areas, 1·54 sq. in., 3·14 sq. in.

Exercises. Page 225.

3. 6·4 sq. cm. 4. 3·7″. 5. 10 cm. 6. 1″.

Exercises. Page 228.

1. 630 sq. cm. 15 cm.

Exercises. Page 231.

2. 8·5 cm. 90°. 3. A circle of rad. 6 cm.
4. 5·20″. 6. 0·25″.

Exercises. Page 235.

1. (i) 16 sq. cm. (ii) 16 sq. cm. 2. (i) 16 sq. cm. (ii) 16 sq. cm.
3. 0·8″. 4. (i) 1·2″. (ii) 12·5 cm. 5. (i) 1·6″, 4·1″. (ii) 3·5 cm.
6. Two concentric circles, radii 2 cm. and 6 cm.

Exercises. Page 237.

1 26″. 2. 48 ft.; 8 ft. 3. 2 cm.; 32 cm.
4. 3·6″. 5. 8100 miles; 10 miles.

Exercises. Page 239.

1. 4 cm. 2. 2·12″. 3. 1·94″. 4. 1·97″.
5. 6·6 cm. 6. 6·6, 2·4. 7. 35·2, 4·8. 8. 3·5 cm.
9. 11·2, 3·2. 10. 9·6, 2·6.

Exercises. Page 241.

1. 2·47″. 2. −3·24″.

Exercises. Page 245.

1. 8, 2. 2. 7, 7. 3. 9·3, 2·7.
4. 9, −4. 5. 11·32, −4·32. 6. 7·24, 2·76.

Exercises. Page 246.

1. 6. 2. 36, 45. 3. 16, 12.
4. (10, $12\frac{1}{2}$); $12\frac{1}{2}$. 5. (17, 18); $12\sqrt{2}=16·97$.
7. 15. 8. 10. 9. Four. (26, 15). 10. 12·84.

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